

No of soln to eqn (non homogeneous, $Ax=b$)

	One / unique soln	∞ / Many soln	No soln
Property	- consistent - independent	- consistent - dependent	not consistent
Graph			
rref	$\left[\begin{array}{ccc c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array} \right]$	$\left[\begin{array}{ccc c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & c \end{array} \right]$ ie $0=c$ not possible	$\left[\begin{array}{ccc c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & 0 \end{array} \right]$ ie $0=0$ always true

No of soln to eqn (homogeneous $AX=0$)
 # such eqns are always consistent

	One / unique soln	∞ / Many soln
Property	- $x_1 = x_2 = x_3 = 0$ is always a soln, it is called trivial soln - If $\det(A) \neq 0$ then trivial soln	- Also called non trivial soln - If $\det(A) = 0$ then non trivial soln - If no of eqn less than no of unknowns, then sys of homogeneous eqn will always have ∞ solns
Graph		
ref	$\left[\begin{array}{ccc c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$ $x_1 = x_2 = x_3 = 0$	$\left[\begin{array}{ccc c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & 0 \end{array} \right]$ $0=0$ always true

for each subspace

- prove subspace
- find subspace

show orthogonal complement

- $N(A) = (C(A^T))^{\perp}$
- $N(A^T) = (C(A))^{\perp}$

	Row space, $C(A^T)$	Left null space, $N(A^T)$
Define	Row space of A = column space of A^T	Left null space of A = Null space of A^T ie $A^T \vec{x} = 0$ take transpose both side $\vec{x}^T A = 0$ ie left null space
Property		
Dim	Dim of row space is also rank	Dim of left null space = #rows - rank
	N dimension	M dimension

Consider $A =$

1	0	-	a	b
0	1	-	-	-
:	:	-	-	-
0	0	-	-	-

Pivot coln

Free coln

$M \times N$

	Null space, $N(A)$	Column space, $C(A)$
Define	Set of vector $\vec{x}$ that satisfy $A\vec{x} = 0$	linear combination of column vectors of A , ie span of column vector of A
Property	only $\vec{0}$ if coln vectors are LI, contain other vectors otherwise - If null space has only $\vec{0}$ then coln vectors are LI and vice versa	- If $N(A) = \vec{0}$, ie column vector are LI they form basis for column space
Dim	- Dimensionality of null space is called nullity, equal to no of free column - nullity = # column - rank	- Dimensionality of column space is rank - rank = # column - nullity
lie in	N dimension	M dimension

Operations

on one

Matrix

Linear system in two unknowns

Takeaway: 3 methods of solving linear system in 2 unknown is substitution, elimination and graphing

Linear systems in two unknowns

$$\begin{aligned} 6x + 3y &= 27 \\ -5x + 2y &= 0 \end{aligned}$$

Substitution

$$\begin{aligned} 6x &= 27 - 3y \\ x &= \frac{9}{2} - \frac{y}{2} = \frac{9}{2} - \frac{5}{2} = \frac{4}{2} \\ -5\left(\frac{9}{2} - \frac{y}{2}\right) + 2y &= 0 \quad \boxed{x=2} \\ -\frac{45}{2} + \frac{5y}{2} + 2y &= 0 \\ -45 + 5y + 4y &= 0 \\ 9y &= 45 \\ \boxed{y=5} \end{aligned}$$

$(x, y) = (2, 5)$

Elimination

$$\begin{aligned} 12x + 6y &= 54 \\ -(15x + 6y) &= 0 \\ \hline -3x &= 54 \\ \boxed{x=2} \\ -5(2) + 2y &= 0 \\ -10 + 2y &= 0 \\ 2y &= 10 \\ \boxed{y=5} \end{aligned}$$

$(x, y) = (2, 5)$

Graphing

$y = -2x + 9$
 $y = \frac{5}{2}x$
 $(x, y) = (2, 5)$

Linear system in 3 unknowns

Takeaway:

- Elimination method to solve
- We do not use graphing method, just the substitution or elimination method.

Linear systems in three unknowns

$$\begin{aligned} [1] \quad 3x - 2y + z &= 2 \\ [2] \quad -x - 5y + 4z &= 1 \quad \checkmark \\ [3] \quad x + 4y - 6z &= -9 \quad \checkmark \end{aligned}$$

Elimination

$$\begin{aligned} -1 - 5(2) + 4(3) &= 1 \\ -1 - 10 + 12 &= 1 \\ -11 + 12 &= 1 \\ 1 &= 1 \end{aligned}$$
$$\begin{aligned} 3(1) - 2(2) + 3 &= 2 \\ 3 - 4 + 3 &= 2 \\ -1 + 3 &= 2 \\ 2 &= 2 \end{aligned}$$

Substitution

$$\begin{aligned} -x - 5y + 4z &= 1 \\ + (x + 4y - 6z) &= -9 \\ \hline -y - 2z &= -8 \\ y + 2z &= 8 \\ \hookrightarrow y &= 8 - 2z \\ 3x - 2y + z &= 2 \\ + (-3x - 15y + 12z) &= 3 \\ \hline -17y + 13z &= 5 \end{aligned}$$
$$\begin{aligned} -17(8 - 2z) + 13z &= 5 \\ -136 + 34z + 13z &= 5 \\ 47z &= 141 \\ \boxed{z=3} \\ y &= 8 - 2(3) \\ y &= 8 - 6 \\ \boxed{y=2} \end{aligned}$$
$$\begin{aligned} x + 4(2) - 6(3) &= -9 \\ x + 8 - 18 &= -9 \\ x - 10 &= -9 \\ \boxed{x=1} \end{aligned}$$

$(x, y, z) = (1, 2, 3)$

Matrix dimensions and entries

Takeaway:

- Ability to find dimensions of given matrix
- Ability to identify elements in matrix
- Matrix is represented by capital letters
- Dimension always represented as row $\times$ column
- column matrix has only one column
- Row matrix has only one row
- Rectangular matrix - where no of row $\neq$ no of column
- Square matrix - that has no of rows = no of column
- Elements of matrix represented as lowercase

Matrix dimensions and entries

$A = \begin{bmatrix} 2 & b \\ -1 & 0 \end{bmatrix}$ 2×2 $a_{1,2} = b$ $a_{2,1} = -1$	$B = \begin{bmatrix} 9 & -2 & 1 & 8 \\ 0 & 0 & -1 & 4 \end{bmatrix}$ 2×4 $b_{2,2} = 0$ $b_{1,4} = 8$ $b_{2,3} = -1$	$C = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$ 3×1 $1 = c_{1,1}$ $2 = c_{3,1}$	$E = [3]$ $D = [-3 \ 4 \ 0 \ b]$ 1×4 $4 = d_{1,2}$ $0 = d_{1,3}$ $b = d_{1,4}$
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Element of matrix represented as (small case name of matrix) row, column

Representing system with matrices

Takeaway:

- How to convert sys of eqns to augmented matrix form
- What is augmented matrix

Representing systems with matrices

$$\begin{aligned} 6x + 3y &= 27 \quad [1] \\ -5x + 2y &= 0 \quad [2] \end{aligned}$$

$$\begin{array}{c} x \quad y \\ 1 \begin{bmatrix} 6 & 3 \end{bmatrix} \quad 6x + 3y \\ 2 \begin{bmatrix} -5 & 2 \end{bmatrix} \quad -5x + 2y \end{array}$$

$$\begin{array}{c} x \quad y \quad c \\ 1 \begin{bmatrix} 6 & 3 & 27 \end{bmatrix} \\ 2 \begin{bmatrix} -5 & 2 & 0 \end{bmatrix} \end{array}$$

$$\begin{aligned} [1] \quad 3x - 2y + z &= 2 \\ [2] \quad -x - 5y + 4z &= 1 \\ [3] \quad x + 4y - 6z &= -9 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ -1 & -5 & 4 & 1 \\ 1 & 4 & -6 & -9 \end{array} \right]$$

$$\begin{aligned} 2y &= 2 - 3x \\ -x - 5y + 4z &= 1 \\ 9 + 6z &= y \end{aligned}$$

$$\begin{aligned} 3x + 2y + 0z &= 2 \\ -x - 5y + 4z &= 1 \\ 0x - y + 6z &= -9 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 2 \\ -1 & -5 & 4 & 1 \\ 0 & -1 & 6 & -9 \end{array} \right]$$

Rearrange to write elements at their proper places in augmented matrix

Augmented matrix: Matrix formed by writing constant terms and augmenting with RHS

Simple row operations

Takeaway:

- 3 row operations
 1. Switching
 2. Scaling (multiplying & dividing)
 3. Adding / subtracting
- & their combinations
- Realize why these operations are equivalent to operations on linear system of eqns

writing linear sys of eqn as augmented matrix

Simple row operations

$$\begin{aligned} [1] \quad 3x - 2y + z &= 2 \\ [2] \quad -x - 5y + 4z &= 1 \\ [3] \quad x + 4y - 6z &= -9 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ -1 & -5 & 4 & 1 \\ 1 & 4 & -6 & -9 \end{array} \right]$$

Switching

$$\begin{aligned} R_1 &\leftrightarrow R_2 & R_2 &\leftrightarrow R_1 \\ \left[\begin{array}{ccc|c} -1 & -5 & 4 & 1 \\ 3 & -2 & 1 & 2 \\ 1 & 4 & -6 & -9 \end{array} \right] \end{aligned}$$

Scaling

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ -3 & -15 & 12 & 3 \\ 1 & 4 & -6 & -9 \end{array} \right]$$

$$3R_2 \rightarrow R_2$$

$$\frac{1}{3}R_2 \rightarrow R_2$$

Adding

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ 0 & -1 & -2 & -8 \\ 1 & 4 & -6 & -9 \end{array} \right]$$

$$R_2 + R_3 \rightarrow R_2$$

$$R_2 - (-R_3) \rightarrow R_2$$

Pivot entries and row echelon forms

Takeaway:

- What are pivots, pivot columns, row echelon form, reduced row echelon form.
- Row operations to take a augmented matrix to row echelon form and reduced row echelon form

Row-echelon: rref

1. Pivots are 1
2. Zero rows at the bottom
3. Staircase pivots

Reduced row-echelon form: rref

4. Pivots alone

$$\left[\begin{array}{ccc|c} -1 & 0 & 2 & 1 \\ 3 & 1 & 1 & -2 \\ 1 & -4 & 0 & 2 \end{array} \right]$$

criteria for a matrix to be ref:

1. All pivot entries are 1
2. zero row if any to bottom
3. pivot entries should form staircase

criteria for matrix to be in rref:

1. matrix should be in ref
2. columns having pivots should not have any other element

Pivot or leading entry of row is first non zero entry in that row

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 3 & 1 & 1 & -2 \\ 1 & -4 & 0 & 2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 7 & 1 \\ 0 & -4 & 2 & 3 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 30 & 7 \end{array} \right]$$

x y z c

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & 7/30 \end{array} \right]$$

Row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & -8/15 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & 7/30 \end{bmatrix}$$

$$\begin{bmatrix} x & y & z & c \\ 1 & 0 & 0 & -8/15 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & 7/30 \end{bmatrix}$$

$x = -8/15$
 $y = -19/30$
 $z = 7/30$

→ reduced row echelon form

As the no of equations increase it will become more and more difficult to use other methods like substitution or elimination to solve for solns.

Gauss - Jordan elimination

Algorithm to transform matrix to reduced row echelon form

Takeaway: Ability to follow Gauss-Jordan elimination steps to convert any matrix to its rref

Gauss-Jordan elimination

- ✓1. Pull scalars
- ✓2. Swap out a 0
- ✓3. First row pivot
- ✓4. Zero out the column
- ✓5. Repeat 2-5

[1] $3x - 2y + z = 2$
[2] $-x - 5y + 4z = 1$
[3] $x + 4y - 6z = -9$

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ -1 & -5 & 4 & 1 \\ 1 & 4 & -6 & -9 \end{array} \right]$$

$R_1 + R_2 \rightarrow R_2$ $R_3 - 3R_1 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ 0 & -1 & -2 & -8 \\ 0 & -14 & 19 & 29 \end{array} \right]$$

$-R_2 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ 0 & 1 & 2 & 8 \\ 0 & -14 & 19 & 29 \end{array} \right]$$

$R_1 - 4R_2 \rightarrow R_1$ $14R_2 + R_3 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -11 & -41 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 47 & 141 \end{array} \right]$$

$(1/47)R_3 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -11 & -41 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$14R_3 + R_1 \rightarrow R_1$ $R_2 - 2R_3 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

x y z C

1. Optional: Pull out any scalars from each row in the matrix.
2. If the first entry in the first row is 0, swap it with another row that has a non-zero entry in its first column. Otherwise, move to step 3.
3. Multiply through the first row by a scalar to make the leading entry equal to 1.
4. Add scaled multiples of the first row to every other row in the matrix until every entry in the first column, other than the leading 1 in the first row, is a 0.
5. Go back step 2 and repeat the process until the matrix is in reduced row-echelon form.

No of solutions

Takeaways:

- Solve sys of eqn
- Pivot column, pivot variable, free column, free variable
- If augmented matrix get converted to a rref that looks like this

One solution
Consistent
Independent
Unique soln

No soln
Inconsistent

Many soln
Consistent
Dependent

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & c \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & 0 \end{array} \right]$$

We can also
say unique
soln

Because this eq
says $0=c$
that is not true

We can also say
that given set of
eqns are
inconsistent

Because this eqn says
 $x=a$, $y=b$ and
 $0z=0$ that is always
true

- Example

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 4 & 4 & -2 & 1 \\ -2 & 2 & -4 & 6 \end{array} \right]$$

$R_2 - 4R_1 \rightarrow R_2$ $2R_1 + R_3 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 0 & 8 & -10 & 13 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

→ solve system of eqns

overdetermined system - when equations are more than unknowns

- It may or may not have a soln

underdetermined system - when unknowns are more than eqns

- Can have no soln or ∞ soln i.e. cannot have unique soln

Operations on two Matrix

Matrix Addition & subtraction

Takeaway:

- How to perform matrix addition and subtraction
- Properties of matrix addition and subtraction

- Matrix addition

$$\begin{bmatrix} 1 & 0 \\ 2 & -4 \end{bmatrix}_{2 \times 2} + \begin{bmatrix} -1 & b \\ 0 & 2 \end{bmatrix}_{2 \times 2} = \begin{bmatrix} 0 & b \\ 2 & -2 \end{bmatrix}_{2 \times 2}$$

- only matrices of same dimensions can be added or subtracted
- matrix addition is commutative and associative

- Matrix subtraction

$$\begin{bmatrix} 1 & 0 \\ 2 & -4 \end{bmatrix} - \begin{bmatrix} -1 & b \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -b \\ 2 & -b \end{bmatrix}$$

- matrix subtraction is not commutative $A - B \neq B - A$
- matrix subtraction is not associative $(A - B) - C \neq A - (B - C)$
- # Same properties as real nos

- commutative means can travel
- associative means can associate

Scalar multiplication

Zero matrix

Takeaway:

- All entries in matrix are 0
- Represented as $O_{4 \times 3}$

Not all 0 matrix are square

Matrix multiplication

Takeaway

- when matrix multiplication is not defined
- dimension of resultant matrix
- steps how to multiply 2 matrices
- properties of matrix multiplication

order of matrices

$$\begin{pmatrix} R \times C \end{pmatrix} \cdot \begin{pmatrix} R \times C \end{pmatrix} = R \times C$$
$$\begin{pmatrix} 2 \times 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \times 2 \end{pmatrix}$$

Matrix multiplication is defined only when no of column of 1st matrix is equal to no of rows of 2nd matrix

Resultant matrix has order = no of row of first matrix $\times$ no of column of 2nd

- steps - how to multiply

$$\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$$

$$2(0) + 3(0) = 0$$

$$2(1) + 3(-1) = -1$$

$$-1(0) + 0(0) = 0$$

$$-1(1) + 0(-1) = -1$$

- properties of matrix multiplication

✗ commutative

✓ Associative

✓ Distributive

$$AB \neq BA$$

$$(A \cdot B)C = A(BC)$$

$$(A+B)C = AC+BC$$

$$C(A-B) = CA-CB$$

// product of real no is commutative
but matrix product is not

Identity matrix

- Representation

I_2

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2×2

I_3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3×3

I_4

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4×4

writing only one subscript is sufficient unlike zero matrix as all identity matrix are square

All diagonal entries are 1, all other entries are 0

Elimination matrix

- What is elimination matrix

Till now we used Gauss-Jordan elimination method to convert a matrix to identity matrix

We can combine those steps in one matrix such that

$$EA = I$$

// Later we will see that E is elimination matrix is same as A^{-1}

- Example:

Q. Find elimination matrix for

$$\begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix}$$

Ans

$$\begin{array}{l} 1. (1/2)R_1 \rightarrow R_1 \quad \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \\ 2. (-1/3)R_2 \rightarrow R_2 \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\ 3. R_1 - 2R_2 \rightarrow R_1 \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \Rightarrow \begin{array}{l} 1. \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \\ 2. \begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix} \\ 3. \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \end{array} \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow E = \begin{bmatrix} 1/2 & 2/3 \\ 0 & -1/3 \end{bmatrix}$$

We would have done these steps to convert matrix to identity matrix

These steps are equivalent to these matrices

Effect of these matrices can be combined by multiplying them in reverse order to get final elimination matrix

- Method 2 - we can also find elimination matrix like we find inverse of matrix

$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & -3 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{1}{3} \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{3} \end{array} \right]$$

Augment with identity matrix

Apply row operations

same elimination matrix

- Dimension of elimination matrix same as org matrix

$$\begin{matrix} E & A \\ 2 \times 2 & 2 \times 2 \end{matrix} = \begin{matrix} I \\ 2 \times 2 \end{matrix} \rightarrow \text{from definition}$$

Elimination matrix can only be found for square matrix

comes out to be same as org matrix

Group Theory

Algebraic structure: A non empty set S is called algebraic structure with respect to binary operation $*$ if $a * b \in S$ where $\forall a, b \in S$

ie $*$ is a closure operation on S

Semigroup: An algebraic structure is called semigroup if it follows associative property ie
 $(a * b) * c = a * (b * c) \quad \forall a, b, c \in S$

Monoid: A semigroup $(S, *)$ is called monoid if there exist an element 'e' $\in S$ such that

$$a * e = e * a = a \quad \forall a \in S$$

element e is called identity element of S with respect to $*$

Group: A monoid $(S, *)$ with identity element 'e' is called a group if for every element $a \in S$, there exist an element $b \in S$ such that

$$(a * b) = (b * a) = e$$

'b' is called inverse of a , denoted as a^{-1}

$$a^{-1} = b$$

also $b^{-1} = a$

- **Abelian group**: Group $(G, *)$ is said to be (commutative gp) abelian if $(a * b) = (b * a) \forall a, b \in G$

- **Summary**

Algebraic str (closure)



Semi gp (associative)



Monoid (identity)



Gp (inverse)



Abelian gp (commutative)

- **Field**: Let F be a non empty set, '+' & '.' are 2 binary operations on F then the system $(F, +, \cdot)$ is called a field if following conditions are satisfied

(i) $(F, +)$ is an abelian group

(ii) $(F', \cdot)$ is an abelian group, where

$$F' = \{x \in F \text{ and } x \neq 0\}$$

(iii) Set F is distributive under binary operation of '+' over '.' i.e

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \forall a, b, c \in F$$

Eg set of rational no
real no
complex no

Matrices

as

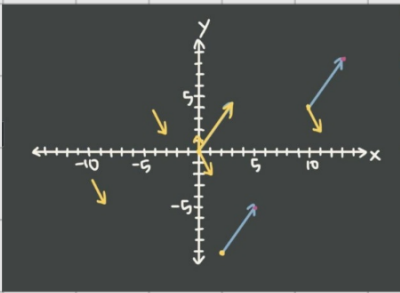
Vectors

Vectors

- What is vector?

1. direction
2. magnitude (length)

- vectors can be shifted



shifting a vector parallelly (even in different plane) does not change its direction and magnitude

- row and vector representation of vector

$$\vec{a} = (3, 4)$$

$$\vec{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

→ column vector

$$\vec{a} = [3 \ 4]$$

→ row vector

- column & row vectors

$$K = \begin{bmatrix} 1 & 0 & 0 & 3 \\ -2 & 1 & 1 & 4 \end{bmatrix}$$

col. vectors

$$k_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$k_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$k_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$k_4 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$\mathbb{R}^2$

→ Col vectors are vectors formed by columns of the matrix

row vectors

$$k_1 = [1 \ 0 \ 0 \ 3]$$

$\mathbb{R}^4$

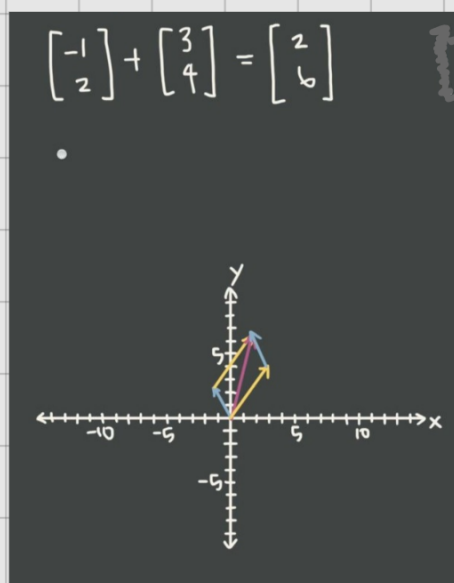
$$k_2 = [-2 \ 1 \ 1 \ 4]$$

→ row vectors are vectors formed by rows of the matrix

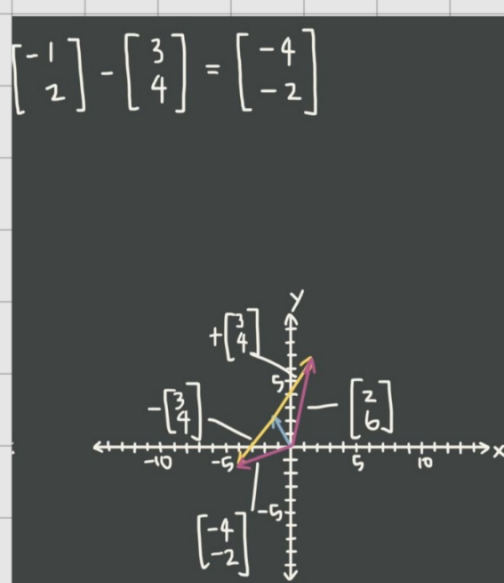
See that column vectors have 2 components, hence they lie in $\mathbb{R}^2$ plane. Similarly there are 2 row vectors, because they are 2 they can be represented in a plane, but they have 4 components hence that plane lies in $\mathbb{R}^4$ space.

Vector operations

vector addition



vector subtraction



vector dot product

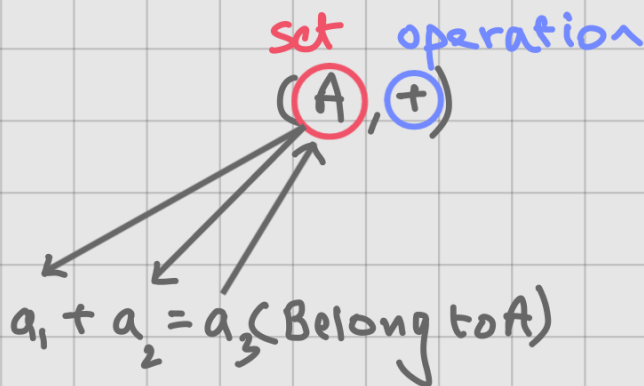
$\vec{a} = (-1, 2) = A$ $\vec{b} = (3, 4) = B$	$\vec{a} \cdot \vec{b} = (-1)(3) + (2)(4)$ $= -3 + 8$ $= 5$	$AB = \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $= (-1)(3) + (2)(4)$ $= -3 + 8 = 5$	$BA = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $= (3)(-1) + (4)(2)$ $= -3 + 8$ $= 5$
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Dot product: represent one as row & other as column matrix

But first matrix has to be row matrix & second column matrix otherwise result will be square matrix

Vector space

Internal & external composition



$+$ is internal composition for A

$$(B, +, \cdot)$$

$a_1 \cdot b_1 = a_2$

// opr for 2nd set
// $a_1, b_1 \in A$

$\cdot$ is external composition for A

Definition for vector space:

Let V be a set, whose elements are called vectors & $(F, +, \cdot)$ be a field whose elements are called scalars. Then set V is called vector space over field F if $V(F)$ satisfy:

- $(V, +)$ is abelian gp with respect to internal composition $+$
- $V(F)$ is closed under scalar multiplication i.e. $(\cdot)$ is external composition i.e. $\forall a \in F, \forall \alpha \in V, a\alpha = \alpha a \in V$
- $(V, +, \cdot)$ satisfy
 - $a\alpha \in V, \forall a \in F \text{ \& } \forall \alpha \in V$
 - $a(\alpha + \beta) = a\alpha + a\beta \quad \forall a \in F \text{ \& } \alpha, \beta \in V$
 - $(a+b)\alpha = a\alpha + b\alpha \quad \forall a, b \in F \text{ \& } \alpha \in V$
 - $a(b\alpha) = (ab)\alpha \quad \forall a, b \in F, \alpha \in V$
 - $1 \cdot \alpha = \alpha$

Properties of Vector space

Let $V(F)$ be a vector space, $\vec{0}$ be the zero and 0 be zero scalar of F , then

- $a\vec{0} = \vec{0}$ $\forall a \in F$
- $0\alpha = \vec{0}$ $\forall \alpha \in V$
- $a(-\alpha) = -(a\alpha)$ $\forall a \in F, \forall \alpha \in V$
- $(-a)\alpha = -(a\alpha)$ $\forall a \in F, \forall \alpha \in V$
- $a(\alpha - \beta) = a\alpha - a\beta$ $\forall a \in F, \forall \alpha, \beta \in V$
- $a\alpha = \vec{0} \Rightarrow \alpha = \vec{0} \text{ or } a = 0$ $\forall a \in F, \forall \alpha \in V$

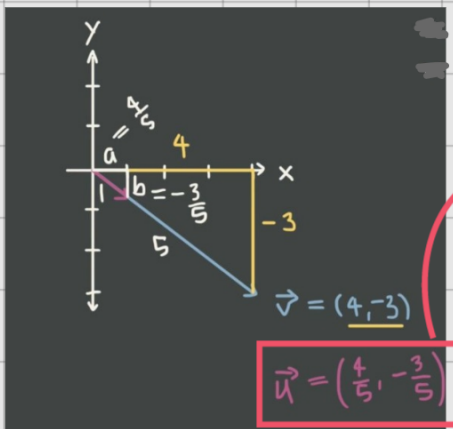
Unit Vector and basis vector

- What is unit vector

✓ direction
✗ length = 1

can be in any direction

- Q Find unit vector along $\vec{v} = (4, -3)$



To find unit vector along a vector divide that vector with its magnitude

$$\begin{aligned}\|\vec{v}\| &= \sqrt{4^2 + (-3)^2} \\ &= \sqrt{16 + 9} \\ &= \sqrt{25} \\ &= 5\end{aligned}$$

- standard vectors

$$\mathbb{R}^2 \quad \begin{aligned}i &= (1, 0) \\ j &= (0, 1)\end{aligned}$$

$$\mathbb{R}^3 \quad \begin{aligned}i &= (1, 0, 0) \\ j &= (0, 1, 0) \\ k &= (0, 0, 1)\end{aligned}$$

Linear combination & span

- Formal definition of linear combination:

Let $\alpha_1, \alpha_2, \dots, \alpha_n \in V(F)$

$a_1, a_2, \dots, a_n \in F$

then linear combination is defined as

$$\sum a_i \alpha_i = \alpha \in V(F)$$

- Linear span

Let $S = \{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n\} \subseteq V(F)$

then linear span of $S = L(S) = \left\{ \sum_{i=1}^n a_i \alpha_i : a_i \in F, \alpha_i \in S \right\}$

- Linear sum of 2 subspaces:

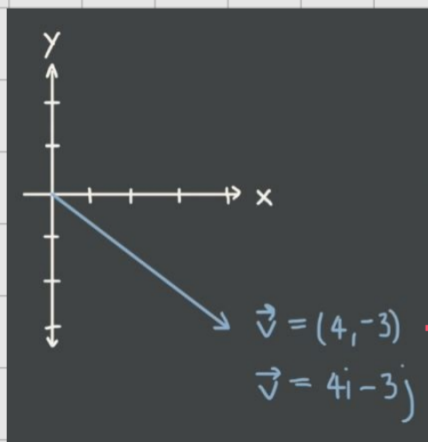
Let W_1 & W_2 be 2 vector subspaces of $V(F)$

the linear sum of W_1 & W_2 is denoted by $W_1 + W_2$ and defined as

$$W_1 + W_2 = \{ \alpha_i + \beta_i : \alpha_i \in W_1 \text{ & } \beta_i \in W_2 \}$$

$W_1 + W_2$ is also a subspace

- What is linear combination of vectors



$\vec{v}$ can be written as sum of scaled (ie linear combination) of $\hat{i}, \hat{j}$ std vectors

$$\vec{v} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It is not compulsory to write $\vec{v}$ as linear combination of std vector. it could have been written as linear combination of any other vectors

$$\begin{aligned} \vec{v} &= 4\hat{i} - 3\hat{j} \\ &= 2\vec{a} + 4\vec{b} - \vec{c} \end{aligned}$$

- What is span of vectors?

Span of vectors is set of all possible linear combination of given vectors

We can write any vector in $\mathbb{R}^2$ as linear combination of $\hat{i}, \hat{j}$ in other words $\mathbb{R}^2 = \text{span}(\hat{i}, \hat{j})$

Again no need to be std vectors or $\perp$ vectors or linear independent vectors. span can be defined for any set of vectors

Linearly independent

- Let $V(F)$ be a vector space and $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be finite set of vectors of $V(F)$ then vectors $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are said to be linearly independent if for scalars $a_1, a_2, \dots, a_n \in F$
$$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0 \Rightarrow a_1 = a_2 = a_3 = \dots = a_n = 0$$
- An infinite set of vectors is linearly independent if its every finite subset is linearly independent

Linearly dependent

- Let $V(F)$ be a vector space and $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be finite set of vectors of $V(F)$ then vectors $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are said to be linearly dependent if there exist scalars $a_1, a_2, \dots, a_n \in F$ such that
$$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0$$

and atleast 1 of $a_1, a_2, \dots, a_n \neq 0$
- # A set of vectors if it contains $\vec{0}$ is linearly dependent

Linear independence in 2D

- We need at least n linearly independent vectors to span $\mathbb{R}^n$ space

$\mathbb{R}^2$: 2, L.I., in $\mathbb{R}^2$

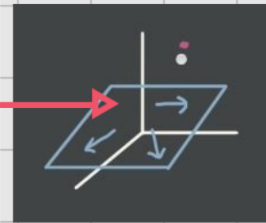
$\mathbb{R}^3$: 3, L.I., in $\mathbb{R}^3$

$\mathbb{R}^n$: n , L.I., in $\mathbb{R}^n$

// If we have more, they will not be linearly independent

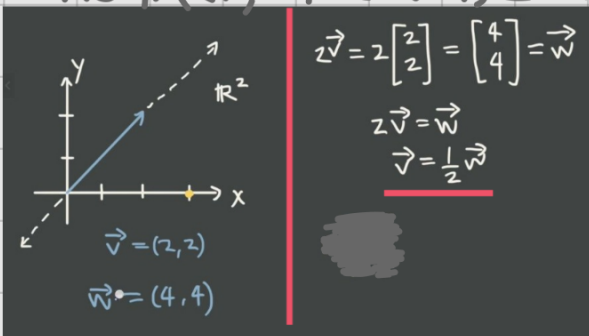
However if they are linearly dependent they will not be able to span $\mathbb{R}^n$

- When 2 two-dimensional vectors lie along the same line (or along parallel lines), they're called collinear, they're linearly dependent, and they won't span $\mathbb{R}^2$.
- When 3 three-dimensional vectors lie in the same plane, they're called coplanar, they're linearly dependent, and they won't span $\mathbb{R}^3$.
- When n n -dimensional vectors lie in the same $(n-1)$ -dimensional space, they're linearly dependent, and they won't span $\mathbb{R}^n$.



Not possible to reach pt

- Vectors are said to be linearly dependent when if they can be written as linear combination of one another, otherwise they are linearly independent



- Test for linear independence / dependence

$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow$ if $c_1 = c_2 = 0$ is only soln then the 2 vectors are linearly independent.

$$c_1 + 0 = 0 \Rightarrow c_1 = 0$$

$$c_1 + bc_2 = 0$$

$$0 + bc_2 = 0$$

$$bc_2 = 0$$

$$c_2 = 0$$

Linear independence in 3D

Check whether these 3 vectors are linearly dependent?

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{ccc|c} c_1 & c_2 & c_3 & \\ \hline 1 & 0 & 1 & 0 \\ 1 & -3 & 0 & 0 \\ 1 & 2 & 3 & 0 \end{array}$$

$$\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 2 & 2 & 0 \end{array}$$

$$\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 4/3 & 0 \end{array}$$

$$\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \end{array}$$

$$\begin{array}{ccc|c} c_1 & c_2 & c_3 & \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}$$

$$\begin{array}{l} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{array}$$

No other combination of c_1, c_2, c_3 can satisfy the eqn
Hence the three vectors are linearly independent

Because these vectors are linearly independent and 3 in no, they span 3D space

Linear subspaces

- what is linear subspace?

A set of vector is said to be subspace if it meet following 3 criteria

Subspace

1. Include zero vector
2. Closed under scalar multiplication
3. Closed under addition

We should be able to take any vector from vector set multiply the vector with scalar, the result must also be in the set

We should be able to take any 2 vectors add them, their result must also be in vector space

See that condition one is contained in condition 2

- check whether the vector set V forms a linear subspace?

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x = -y, y = -x \right\}$$
$$= \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \dots \right\}$$

✓ Scalar multiplication

$$v = \begin{bmatrix} x \\ -x \end{bmatrix} \quad \begin{bmatrix} -y \\ y \end{bmatrix}$$

$$c \begin{bmatrix} x \\ -x \end{bmatrix} = \begin{bmatrix} cx \\ -cx \end{bmatrix} \quad \begin{array}{l} cx + (-cx) = 0 \\ cx - cx = 0 \\ 0 = 0 \checkmark \end{array}$$

$$c \begin{bmatrix} -y \\ y \end{bmatrix} = \begin{bmatrix} -cy \\ cy \end{bmatrix} \quad \begin{array}{l} -cy + cy = 0 \\ 0 = 0 \checkmark \end{array}$$

✓ Addition

$$\begin{bmatrix} x \\ -x \end{bmatrix} + \begin{bmatrix} -y \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ -x+y \end{bmatrix} = \begin{bmatrix} x-y \\ y-x \end{bmatrix}$$
$$x-y + y-x = 0 \quad 0 = 0 \checkmark$$

Because V vector set satisfy the 3 conditions it is a linear subspace

- **Theorem-1:** Intersection of subspaces is subspace
- **Theorem-2:** Union of 2 subspaces may not be subspace

- Possible subspaces

In fact, there are three possible subspaces of $\mathbb{R}^2$.

1. $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.
2. Any line through the origin $(0,0)$ is a subspace of $\mathbb{R}^2$.
3. The zero vector $\vec{0} = (0,0)$ is a subspace of $\mathbb{R}^2$.

Similarly, there are four possible subspaces of $\mathbb{R}^3$.

1. $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.
2. Any line through the origin $(0,0,0)$ is a subspace of $\mathbb{R}^3$.
3. Any plane through the origin $(0,0,0)$ is a subspace of $\mathbb{R}^3$.
4. The zero vector $\vec{0} = (0,0,0)$ is a subspace of $\mathbb{R}^3$.

And we could extrapolate this pattern to get the possible subspaces of $\mathbb{R}^n$, as well.

Spans are always subspace

- show spans are subspaces

$$V = \text{Span} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right) \rightarrow \text{plane}$$
$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

// By definition spans are all possible linear combination of set of vectors (by definition) Hence they follow the 3 condition for being subspace

- In general all spans, be it 2 vector, 3 vector linearly dependent vector whatever are subspaces

Basis of Subspace

formal definition: Let $V(F)$ be the vector space over the field F . the subset $S \subseteq V(F)$ is called Basis of $V(F)$ if

1. S is set of LI vectors
2. $L(S) = V(F)$

Dimension of Subspace

Let $B = \text{Basis for } V(F)$ vector space then
dimension of $V(F) = \dim V(F) = \dim V = \text{no of element in } B$

Dim of sum of Subspaces

let W_1 and W_2 are subspaces of finite dimensional vector space $V(F)$ then

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

Existence theorem

• There exists (at least one) basis for every finite dimension vector space

• **Proof** - $L(V(F)) = V(F)$

If $V(F) = (a_1, a_2, \dots, a_n)$

either they are LI then theorem conclude otherwise keep removing a till we get LI set

Dimension theorem

No of elements in all possible basis for finite dimension space is same

Extension theorem

Every LI subset of finite dimension vector space forms a part of basis for $V(F)$

Theorem

Each set of $(n+1)$ or more vectors of finite dim vector space $V(F)$ of $\dim(V) = n$ is linearly dependent

Proof: Let $S = \{s_1, s_2, \dots, s_{n+1}\}$ be LI then by extension theorem we can even more vectors to make it basis this is in contradiction to dimension theorem (proof by contradiction)

Theorem

- Let $\dim V = n$ then any set of n LI vectors in V forms a basis of V
- Proof: Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be LI set if $L(S) = V$ then S is basis (it was already LI otherwise we can add LI vector to form basis (extension theorem)
The basis formed will have more than n vectors contradict previous theorem.

Theorem

- If $\dim(V) = n$ and S is any set containing n vectors that gen V then S is also basis
- Proof: given $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ & $L(S) = V$
If S is LI then proof is complete otherwise we can remove a LD vector without affecting its span, till all remaining vectors are LI and it still gen V that contradict

previous theorem

Theorem

Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be basis of $V(F)$ with $\dim(V) = n$ then every $\alpha \in V(F)$ is uniquely expressed as $\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$ $a_i \in F$

Proof: Let there exist two combinations

$$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = b_1\alpha_1 + b_2\alpha_2 + \dots + b_n\alpha_n$$

$$= (a_1 - b_1)\alpha_1 + (a_2 - b_2)\alpha_2 + \dots + (a_n - b_n)\alpha_n = 0$$

but $\alpha_1, \alpha_2, \dots, \alpha_n$ are LI

Theorem - dim of subspace

If $\dim(V) = n$ and $W \subseteq V$ is subspace then $\dim(W) \leq n$

Dot and Cross Product

Dot product

- How vector dot product is calculated

$$\vec{a} = (a_1, a_2, a_3)$$

$$\vec{b} = (b_1, b_2, b_3)$$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

// Just multiply corresponding entries

- Dot product can also be found by using matrix product

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\vec{a} \cdot \vec{b} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

- Dot product of a vector with itself is equal to square of its magnitude

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$$

$$\boxed{\|\vec{a}\|^2} = a_1^2 + a_2^2 = \boxed{\vec{a} \cdot \vec{a}} = a_1 a_1 + a_2 a_2 = a_1^2 + a_2^2$$

- Properties of dot product

1. Commutative

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

2. Distributive

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

3. Associative

$$(3\vec{r}) \cdot \vec{s} = 3(\vec{r} \cdot \vec{s})$$

Cauchy - Schwarz inequality

- what is Cauchy-Schwarz inequality?

$$|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \|\vec{b}\|$$

- equal when the two vectors are parallel i.e. colinear i.e. linearly dependent
- less than otherwise i.e. when the 2 vectors are linearly independent
- This inequality can also be used to check if the 2 vectors are linearly dependent or independent

$$i = (1, 0) \quad j = (0, 1)$$

$$\begin{aligned}\|i\| &= \sqrt{1^2 + 0^2} \\ &= \sqrt{1+0} \\ &= \sqrt{1} \\ &= 1\end{aligned}$$

$$\begin{aligned}i \cdot j &= (1)(0) + (0)(1) \\ &= 0+0 \\ &= 0\end{aligned}$$

$$|i \cdot j| = 0$$

$$0 \leq 1 \cdot 1$$

$$0 < 1 \quad \underline{\text{L.I.}}$$

$$\vec{a} = (2, 1) \quad \vec{b} = (-6, -3)$$

$$\begin{aligned}\|\vec{a}\| &= \sqrt{2^2 + 1^2} \\ &= \sqrt{4+1} \\ &= \sqrt{5}\end{aligned}$$

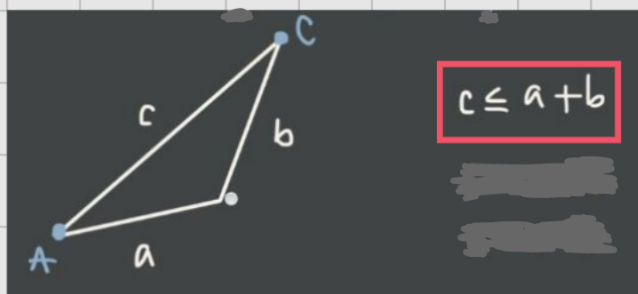
$$\begin{aligned}\|\vec{b}\| &= \sqrt{(-6)^2 + (-3)^2} \\ &= \sqrt{36+9} \\ &= \sqrt{45} = 3\sqrt{5}\end{aligned}$$

$$\begin{aligned}|\vec{a} \cdot \vec{b}| &= |(2)(-6) + (1)(-3)| \\ &= |-12-3| \\ &= |-15| \\ &= 15 = \|\vec{a}\| \|\vec{b}\|\end{aligned}$$

Hence LD i.e.
 $\vec{b} = 2\vec{a}$

Vector triangle inequality

- What is vector triangle inequality?

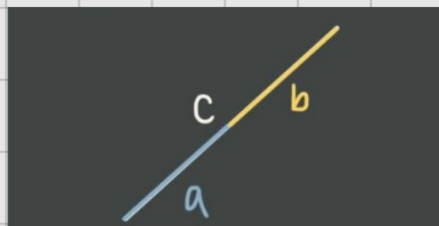


// Shortest dist b/w 2 points is straight line joining them

- It can be written in vector form

$$\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$

- Equality holds when 2 vectors are colinear i.e. linearly dependent



- This can be used to test if 2 vectors are linearly dependent or independent

$$\text{Ex: } \vec{a} = (1, 1, 2, 2) \\ \vec{b} = (1, 0, -3, 1)$$

$$\|\vec{a}\| = \sqrt{1^2 + 1^2 + 2^2 + 2^2} \\ = \sqrt{1 + 1 + 4 + 4} \\ = \sqrt{10}$$

$$\|\vec{b}\| = \sqrt{1^2 + 0^2 + (-3)^2 + 1^2} \\ = \sqrt{1 + 9 + 1} \\ = \sqrt{11}$$

$$\|\vec{a} + \vec{b}\| = \sqrt{2^2 + 1^2 + (-1)^2 + 3^2} \\ = \sqrt{4 + 1 + 1 + 9} \\ = \sqrt{15}$$

$$\sqrt{15} \leq \sqrt{10} + \sqrt{11} \\ 3.87 < 3.16 + 3.32$$

Hence they are linearly independent

Prove Vector triangle inequality

$$|\vec{x} + \vec{y}|^2 = (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y})$$

$$= \vec{x} \cdot \vec{x} + \vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{x} + \vec{y} \cdot \vec{y}$$

$$= |\vec{x}|^2 + |\vec{y}|^2 + \underline{2 \vec{x} \cdot \vec{y}}$$

$$\leq |\vec{x}|^2 + |\vec{y}|^2 + \underline{2 |\vec{x}| |\vec{y}|}$$

cauchy-schwarz
inequality

$$|\vec{x} + \vec{y}|^2 \leq (|\vec{x}| + |\vec{y}|)^2$$

$$|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$$

Angle between two vectors

- vector dot product

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$$

- when $\vec{a} \perp \vec{b}$, $\cos \theta = \cos 90^\circ = 0$, in other words if $\vec{a} \cdot \vec{b} = 0$ and $|\vec{a}| \neq 0$, $|\vec{b}| \neq 0$ then $\vec{a} \perp \vec{b}$ this even applies in higher dimensions

$$\left. \begin{array}{l} \vec{a} = (1, 0, -2, 0) \\ \vec{b} = (0, 3, 0, 1) \end{array} \right\} \mathbb{R}^4$$

$$\begin{aligned} \vec{a} \cdot \vec{b} &= (1)(0) + (0)(3) + (-2)(0) + (0)(1) \\ &= 0 + 0 + 0 + 0 \\ &= 0 \end{aligned}$$

// for higher dimension we say the vectors are orthogonal

- we can also use the formula to find angle b/w any 2 vectors

$$\vec{a} = (0, 3) \quad \vec{b} = (-1, -4)$$

$$\begin{aligned} \vec{a} \cdot \vec{b} &= (0)(-1) + (3)(-4) \\ &= 0 - 12 \\ &= -12 \end{aligned}$$

$$\begin{aligned} \|\vec{a}\| &= \sqrt{0^2 + 3^2} \\ &= \sqrt{9} \\ &= 3 \end{aligned}$$

$$\begin{aligned} \|\vec{b}\| &= \sqrt{(-1)^2 + (-4)^2} \\ &= \sqrt{1 + 16} \\ &= \sqrt{17} \end{aligned}$$

$$-12 = 3\sqrt{17} \cos \theta$$

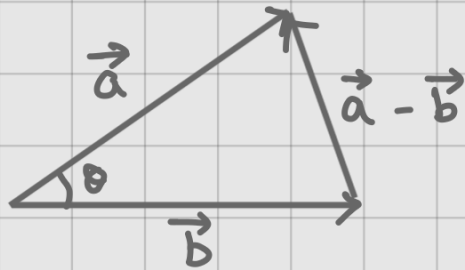
$$\cos \theta = -\frac{12}{3\sqrt{17}} = -\frac{4}{\sqrt{17}}$$

$$\begin{aligned} \arccos(\cos \theta) &= \arccos\left(-\frac{4}{\sqrt{17}}\right) \\ \theta &= 166^\circ \end{aligned}$$

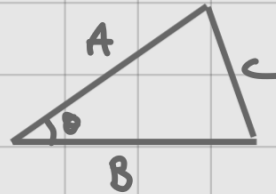
→ Always gives the smaller angle

Formula for angle b/w 2 vector

- consider



Apply law of cosines ie



$$C^2 = A^2 + B^2 - 2AB \cos \theta$$

$$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos \theta$$

$$(\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos \theta$$

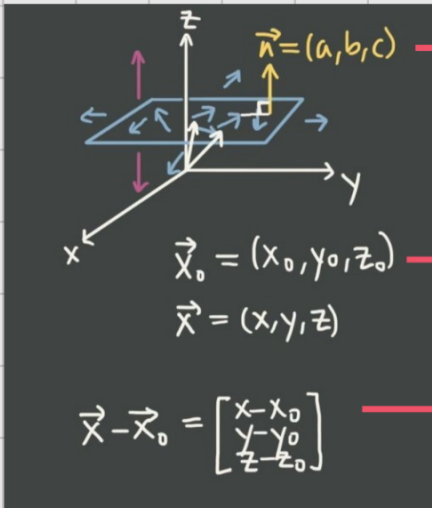
$$\cancel{\vec{a} \cdot \vec{a}} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \cancel{\vec{b} \cdot \vec{b}} = \cancel{|\vec{a}|^2} + \cancel{|\vec{b}|^2} - 2|\vec{a}||\vec{b}|\cos \theta$$

$$-2\vec{a} \cdot \vec{b} = -2|\vec{a}||\vec{b}|\cos \theta$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos \theta$$

Eqn of plane & normal vectors

- Eqn of plane



vector $\perp$ to plane

Pt on plane

General vector in plane

$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$$

$$[a \ b \ c] \begin{bmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{bmatrix} = 0$$

$$[a(x-x_0) + b(y-y_0) + c(z-z_0) = 0]$$

General vector & normal vector are $\perp$

Eqn of plane

- Question

normal vector

$$\vec{n} = (2, 5, -3)$$

$a \ b \ c$

pt on plane

$$(1, 0, -2)$$

$x_0 \ y_0 \ z_0$

$$2(x-1) + 5(y-0) - 3(z+2) = 0$$

$$2x - 2 + 5y - 3z - 6 = 0$$

$$2x + 5y - 3z = 8$$

$$Ax + By + Cz = D$$

$\vec{n} = (A, B, C)$

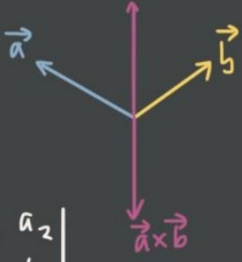
$\vec{n} = (2, 5, -3)$

normal vector is just coeff of x, y, z in eqn of plane

Cross product

- How to find cross product of 2 vectors using determinants?

Cross products

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= \mathbf{i}(a_2 b_3 - a_3 b_2) - \mathbf{j}(a_1 b_3 - a_3 b_1) + \mathbf{k}(a_1 b_2 - a_2 b_1)\end{aligned}$$


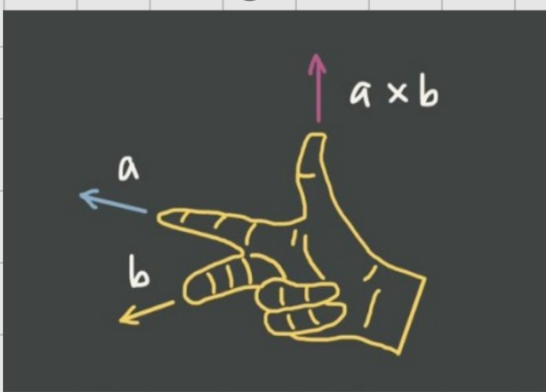
- Example

$$\begin{aligned}\vec{a} &= (1, 0, 2) \\ \vec{b} &= (-2, 1, 0) \\ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ -2 & 1 & 0 \end{vmatrix} &= \mathbf{i} \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} \\ &= \mathbf{i}(0 \cdot 0 - 2 \cdot 1) - \mathbf{j}(1 \cdot 0 - 2(-2)) + \mathbf{k}(1 \cdot 1 - 0(-2)) \\ &= \mathbf{i}(-2) - \mathbf{j}(4) + \mathbf{k}(1) \\ &= -2\mathbf{i} - 4\mathbf{j} + \mathbf{k} \\ &= (-2, -4, 1) = \vec{a} \times \vec{b}\end{aligned}$$

- Length of cross product

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta$$

- How to find direction of resultant?

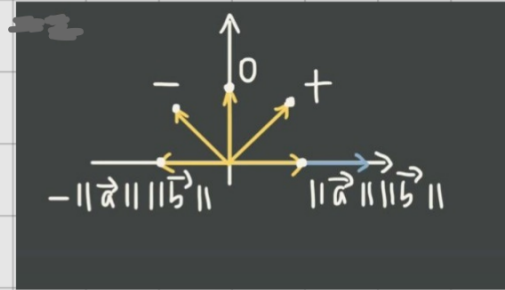


Dot and Cross Product as opposite ideas

- How dot product depend on direction of 2 vectors?

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$$

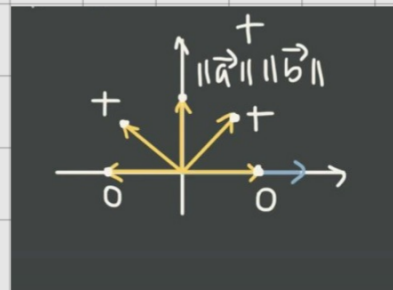
$\uparrow$ For $\theta = 90^\circ$: $\vec{a} \cdot \vec{b} = 0$
 $\rightarrow$ For $\theta = 0^\circ$: $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\|$
 $\leftarrow$ For $\theta = 180^\circ$: $\vec{a} \cdot \vec{b} = -\|\vec{a}\| \|\vec{b}\|$



- How cross product depend on direction of 2 vectors?

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta$$

For $\theta = 90^\circ$: $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\|$
 $\rightarrow$ For $\theta = 0^\circ$: $\|\vec{a} \times \vec{b}\| = 0$
 $\leftarrow$ For $\theta = 180^\circ$: $\|\vec{a} \times \vec{b}\| = 0$



Matrix Vector Product

Multiplying Matrix with vector

Matrix x vector

Matrix $\leftarrow A = \begin{bmatrix} 1 & 0 \\ -2 & 4 \\ 0 & 1 \end{bmatrix}$

Vector $\leftarrow \vec{b} = (5, -1)$

$A\vec{b} = \begin{bmatrix} 1 & 0 \\ -2 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

$(3 \times 2) \times (2 \times 1) = 3 \times 1$

$= \begin{bmatrix} 1(5) + 0(-1) \\ -2(5) + 4(-1) \\ 0(5) + 1(-1) \end{bmatrix}$

$A\vec{b} = \begin{bmatrix} 5 \\ -14 \\ -1 \end{bmatrix}$

$\xrightarrow{A\vec{b}}$ matrix x vector

$\rightarrow$ result will always have 1 column

vector x matrix

$\vec{w}B$

$\vec{w}B = [1 \ 0 \ -2] \begin{bmatrix} 3 & -4 \\ 1 & 0 \\ 0 & -2 \end{bmatrix}$

$(1 \times 3) \quad (3 \times 2)$

$= [(1)(3) + (0)(1) - 2(0) \quad (1)(-4) + 0 + 4]$

$\vec{w}B = [3 \ 0]$

Null space and $Ax=0$

- Null space of matrix A is set of vectors that

$\vec{x}$ That satisfies
 $A\vec{x} = \vec{0}$
is in the null space
 $N(A)$

$\rightarrow$ How null space is represented

- Null space is subspace - it satisfies the 3 conditions to be subspace

✓ 1. zero vector

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{1,1}(0) + a_{1,2}(0) \\ a_{2,1}(0) + a_{2,2}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

✓ 2. under addition

$$A\vec{x}_1 = \vec{0}, A\vec{x}_2 = \vec{0} \implies A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0} + \vec{0} = \vec{0}$$

✓ 3. under multiplication

$$A(c\vec{x}_1) = c(A\vec{x}_1) = c\vec{0} = \vec{0}$$

- Example - Find null space of matrix A

$$A = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & | & 0 \\ 2 & -1 & | & 0 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 4 & -2 & | & 0 \\ 2 & -1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \xrightarrow{x_1 = \frac{1}{2}x_2} \begin{bmatrix} 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$x_1 - \frac{1}{2}x_2 = 0 \implies x_1 = \frac{1}{2}x_2$$

$$N(A) = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \dots \right\}$$

→ vector set of all such vectors

- Linear independence and null space

- Null space contains only $\vec{0}$ if the column vectors are linearly independent and contain other vectors if column vectors are linearly dependent

$$N(A) = \vec{0} \quad \text{L.I.}$$

$$N(A) = \vec{0}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{L.D.}$$

// Because the above eqn can also be written like this

$$x_1 \begin{bmatrix} 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which is also test for linear independence

- Vice versa if null space contains only $\vec{0}$ we can say that column vectors are linearly independent and if null space contain more than $\vec{0}$ then column vectors are linearly dependent

Null space of matrix

- Find null space of matrix A

$$A = \begin{bmatrix} 2 & 0 & -1 & 2 \\ -4 & 0 & 2 & -4 \\ -6 & 0 & -3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A \quad \vec{x} = \vec{0}$

$(3 \times 4) \quad (4 \times 1) = 3 \times 1$

$R_2 + 2R_1 \rightarrow R_2$

$$A = \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ -6 & 0 & -3 & -6 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 0 & -1 & 2 \\ -6 & 0 & -3 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$3R_1 + R_2 \rightarrow R_2$

$$= \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & -1/2 & 1 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

→ will be column vector

→ so that matrix multiplication is defined

→ Not writing complete augmented matrix as the complete column is 0 only

$$= \begin{bmatrix} 1 & 0 & -1/2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 + \frac{1}{2}R_2 \rightarrow R_1$$

$$\text{rref}(A) = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

pivot free

$$x_1 + x_4 = 0 \rightarrow x_1 = -x_4 + 0x_2$$

$$x_3 = 0 \rightarrow x_3 = 0 + 0x_4$$

$$x_4 = -x_1 + 0x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$N(A) = \text{span} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

→ writing pivot variables in terms of free variables

→ same as span

Column space and $Ax = b$

- what is column space of matrix A

$$C(A) = \text{linear combinations of columns of } A$$

$$= \text{span of columns of } A$$

- for example

$$A = \begin{bmatrix} 2 & 4 \\ 3 & b \end{bmatrix} \quad C(A) = \text{span} \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ b \end{bmatrix} \right)$$

// Because it is defined as span it is also a subspace

- In other words

$$A \vec{x} = \vec{b}$$

$$\begin{bmatrix} 2 & 4 \\ 3 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b \\ q \end{bmatrix}$$

If $\vec{b} \in C(A)$,
Then $\vec{x}$ exists
as a solution

$$x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ b \end{bmatrix} = \begin{bmatrix} b \\ q \end{bmatrix}$$

- If column vectors of A are linearly independent ($N(A) = \vec{0}$) then they form basis for vector space
- If column vectors of A are not linearly independent they do not form the basis for the column space

$N(A) = \vec{0}$ columns of A L.I. $\rightarrow$ Basis
 $N(A) = \vec{0}, \vec{v}_1$ " " " L.D. $\rightarrow$ Do not form a basis

- Example - consider

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \quad C(A) = \text{span} \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix} \right)$$

- findings its null space

$$\begin{aligned}
 \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} &\xrightarrow{\text{rref}(A)} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
 x_1 + 2x_2 &= 0 \quad \text{pivot} \quad \text{free} \\
 x_1 &= -2x_2 \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\
 N(A) &= N(\text{rref}(A)) = \text{span} \left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)
 \end{aligned}$$

Hence column vectors are not linearly independent

- We can remove independent column vector to remove redundancy so that remaining set of vectors are linearly independent and form basis

Basis

$$x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = 0$$

$$\text{Basis for } C(A) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$C(A) = \text{span} \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \right)$$

Solving $A\vec{x} = \vec{b}$

- complementary, particular & gen soln to $A\vec{x} = \vec{b}$

$$\begin{array}{ll} A\vec{x} = \vec{0} & N(A) \quad \text{complementary solution} \\ A\vec{x} = \vec{b} & C(A) \quad \text{particular solution} \end{array}$$

- Mathematically the soln are represented as

$$\begin{array}{l} A\vec{x}_n = \vec{0} \\ A\vec{x}_p = \vec{b} \end{array}$$

- Gen soln is sum of both the soln

$$\begin{array}{l} A\vec{x}_n + A\vec{x}_p = \vec{0} + \vec{b} \\ A(\vec{x}_n + \vec{x}_p) = \vec{b} \\ \vec{x} = \vec{x}_n + \vec{x}_p \quad \text{general solution} \end{array}$$

- Question

Find the general solution to $A\vec{x} = \vec{b}$.

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 2 & 1 & 4 \\ 3 & 4 & 4 & 7 \end{bmatrix} \quad \text{with } \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

- first find complementary soln then find particular soln
- write gen (complete) soln as their sum

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Dimensionality, nullity & rank

- Dimensionality of a vector space refers to no of basis vectors reqd to span that space

- Nullity and rank

- consider a matrix

$$A = \begin{bmatrix} 2 & 0 & 4 & -2 \\ 1 & 3 & -1 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

Convert to reduced row echelon form

$$A = \begin{matrix} & x_1 & x_2 & x_3 & x_4 \\ \begin{matrix} m \times n \\ 3 \times 4 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 7/3 \\ 0 & 1 & 0 & -4/3 \\ 0 & 0 & 1 & -5/3 \end{bmatrix} \end{matrix}$$

- Nullity is dimensionality of null space

Dimension: Null space

"Nullity"

$\dim(N(A))$, nullity(A)

$$\text{nullity}(A) = 1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -7/3 \\ 4/3 \\ 5/3 \\ 1 \end{bmatrix}$$

$$N(A) = \text{span}(\uparrow)$$

→ equal to no of free column

- Rank is dimensionality of column space

Dimension: Column space

"Rank"

$\dim(C(A))$, rank(A)

$$\text{rank}(A) = 3$$

$$C(A) = \text{span}(\ast)$$

Equal to no of pivot columns

Rank nullity theorem

- Let $T: U \rightarrow V$ be linear transformation from U to V where U & V are finite dimension vector spaces then

$$\text{Rank}(T) + \text{Nullity}(T) = \text{dimension of } U$$

• Proof:

- Let $\dim(U) = n$
- $N(T)$ is subspace of U , its dimension is $k \leq n$ and its basis be $S = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$
- By extension theorem S can be extended to $S' = \{\alpha_1, \alpha_2, \dots, \alpha_k, \alpha_{k+1}, \dots, \alpha_n\}$ such that S' is basis of U

- the image $\{T(\alpha_1), T(\alpha_2) \dots T(\alpha_k) \dots T(\alpha_n)\}$ belong to range $R(T)$
- we will show $T(\alpha_{k+1}), T(\alpha_{k+2}) \dots T(\alpha_n)$ form basis for $R(T)$, to show they form basis

1. Their linear span is $R(T)$

If $\beta \in R(T)$ then it must have a preimage $\alpha \in U$ such that $\beta = T(\alpha)$

$$\beta = T(a_1\alpha_1 + a_2\alpha_2 \dots a_k\alpha_k \dots a_n\alpha_n)$$

$$\beta = a_1T(\alpha_1) + a_2T(\alpha_2) \dots a_kT(\alpha_k) \dots a_nT(\alpha_n)$$

$$\beta = a_1 \cdot 0 + a_2 \cdot 0 \dots a_{k+1}T(\alpha_n) \dots a_nT(\alpha_n)$$

$$\beta = a_{k+1}T(\alpha_n) + \dots + a_nT(\alpha_n)$$

Any $\beta \in R(T)$ can be expressed as LI

2. They are LI

Let there exist $a_{k+1}, a_{k+2} \dots a_n \in F$ such that

$$a_{k+1}T(\alpha_{k+1}) + a_{k+2}T(\alpha_{k+2}) \dots = 0$$

$$T(a_{k+1}\alpha_{k+1} + a_{k+2}\alpha_{k+2} \dots) = 0 \quad \text{if } T \text{ is linear}$$

$\Rightarrow a_{k+1}\alpha_{k+1} + a_{k+2}\alpha_{k+2} \dots a_n\alpha_n \in N(T)$ hence can be expressed as linear combination of basis of $N(T)$

$$a_{k+1}\alpha_{k+1} \dots a_n\alpha_n = c_1\alpha_1 + c_2\alpha_2 \dots c_k\alpha_k$$

$$-c_1\alpha_1 - c_2\alpha_2 - \dots - c_k\alpha_k + a_{k+1}\alpha_{k+1} - \dots - a_n\alpha_n = 0$$

since $\alpha_1, \alpha_2, \dots, \alpha_n$ form a basis for U so they are LI

• summary of proof

$$(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_k, \alpha_{k+1}, \dots, \alpha_n)$$

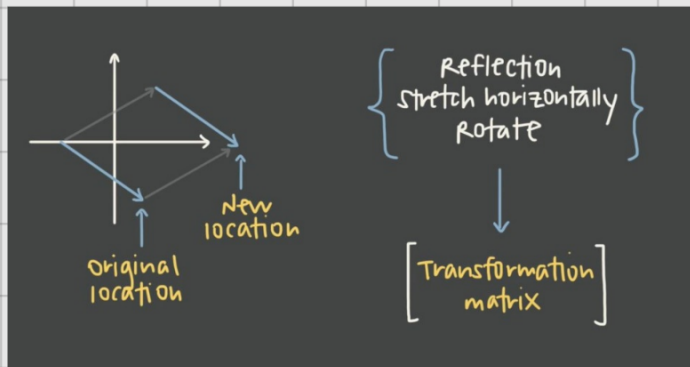
Basis of null space

By extension theorem, basis of U

Prove basis of range

Transformations

Intro



We are interested because these (reflection, stretching and rotation) can be encompassed in a matrix

Transformations are like functions

- Transformations are like functions

function $f(x) = x^2$

x	$f(x)$
0	0
1	1
2	4
3	9

vector-valued function

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_2 \\ 3x_1 \end{bmatrix}$$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$
$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \end{bmatrix}$
$\begin{bmatrix} -5 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 7 \\ -15 \end{bmatrix}$

- But we do not call them vector valued function we call them transformation and represent them as $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_2 \\ 3x_1 \end{bmatrix}$
- we use word transformation instead of function to suggest transformation of vector in space

Linear transformation

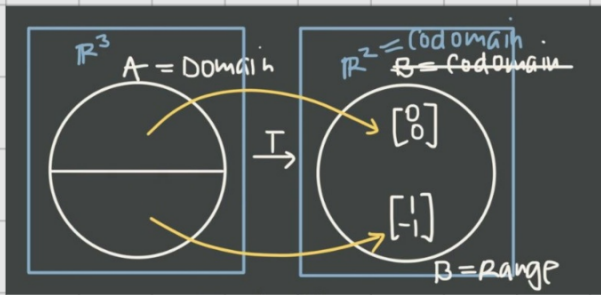
Let U and V be two vector spaces.
Then mapping $J: U \rightarrow V$ is called linear transformation if

$$J(x+b) = J(x) + J(b)$$

$$J(ax) = aJ(x)$$

where x, b are vector in $U(F)$ and a is any scalar

$$T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$$



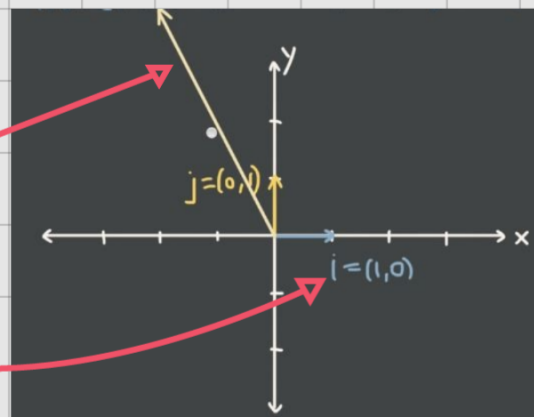
Domain: $\mathbb{R}^3$
 Codomain: $\mathbb{R}^2$
 Range: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

Transformation Matrices and image of subset

Transformation matrix

- Tells us where std basis vector will go

$$T = \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}$$



- Similarly in 3D

$$T = \begin{bmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

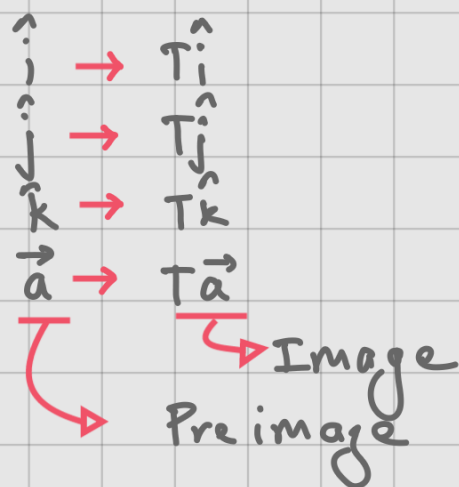
$i: (1, 0, 0) \rightarrow (1, 1, 1)$
 $j: (0, 1, 0) \rightarrow (2, 2, 2)$
 $k: (0, 0, 1) \rightarrow (3, 3, 3)$

- We can use transformation matrix to find new loc of any vector

$$T\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) : \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

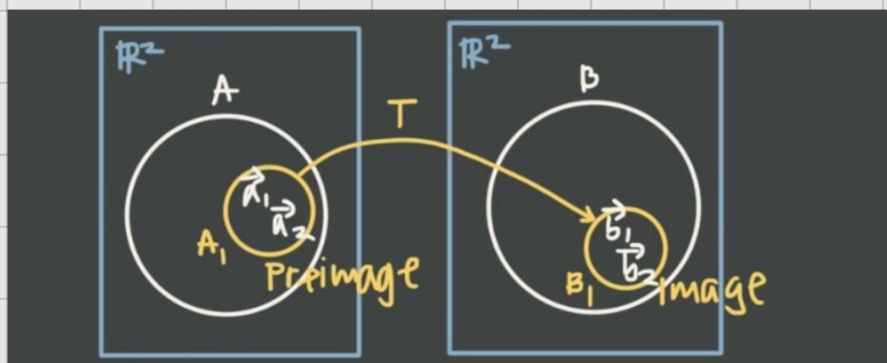
$$= \begin{bmatrix} 3(1) - 2(-2) \\ 2(1) + 4(-2) \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$$

- Transformation is represented as



Preimage, image and kernel

- Pre image and image



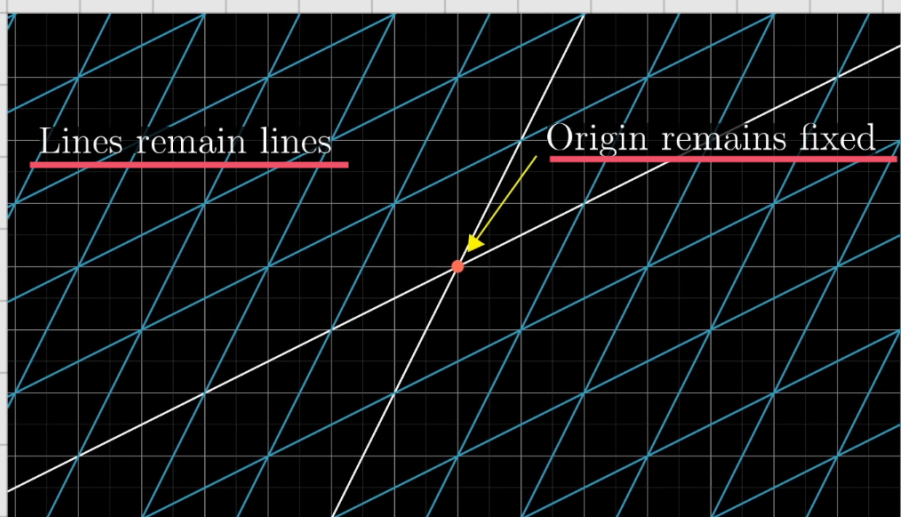
It is also possible to find preimage if image vector is given with a transformation matrix $A\vec{x} = \vec{c}$, find $\vec{x}$

- kernel - all vectors in A that are mapped to $\vec{0}$ in B same as null space

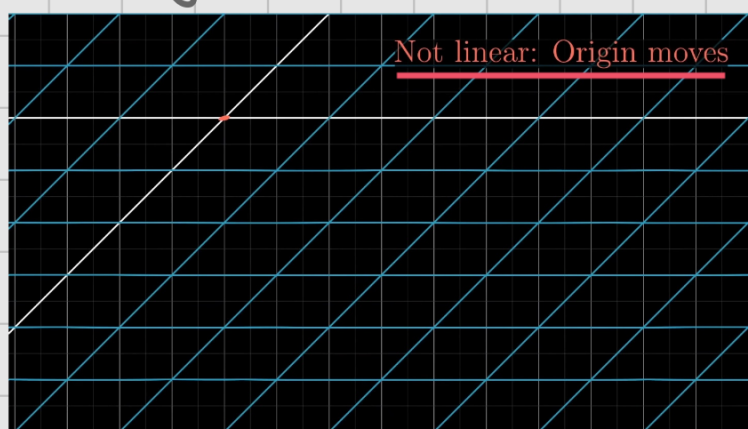
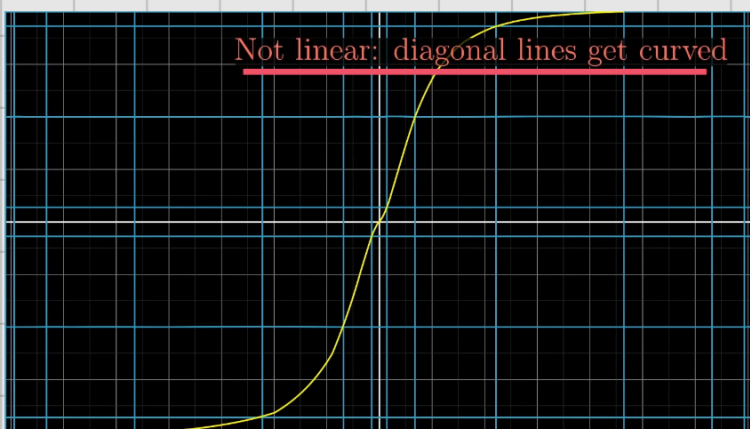
$\text{Ker}(T)$: all the vectors that map to $\vec{0}$

Dimension of kernel is called nullity

- visually linear transformation have 2 basic conditions



Hence this is not a linear transformation



Rotation matrix

- Tx matrix if rotation being performed in 2D

If we're rotating in $\mathbb{R}^2$, we're rotating counterclockwise around the origin through the angle θ , and the transforming rotation matrix will be

$$\text{Rot}_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

And the transformation to rotate any vector $\vec{x}$ in $\mathbb{R}^2$ will be

$$\text{Rot}_\theta(\vec{x}) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Transformation matrix if rotation in 3D

If we're rotating in $\mathbb{R}^3$, the transforming rotation matrix will be different depending on which axis we're rotating around.

$$\text{Rot}_\theta \text{ around } x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

$$\text{Rot}_\theta \text{ around } y = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$\text{Rot}_\theta \text{ around } z = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And the transformation to rotate any vector $\vec{x}$ in $\mathbb{R}^3$ will be

$$\text{Rot}_\theta \text{ around } x(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\text{Rot}_\theta \text{ around } y(\vec{x}) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\text{Rot}_\theta \text{ around } z(\vec{x}) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Rotation is also linear transformation, hence follows properties of linear transformation

$$\begin{aligned} \text{Rot}_\theta(\vec{a} + \vec{b}) &= \text{Rot}_\theta(\vec{a}) + \text{Rot}_\theta(\vec{b}) \\ \text{Rot}_\theta(c\vec{a}) &= c \text{Rot}_\theta(\vec{a}) \end{aligned}$$

In 2D plane if we rotate by 180° then -1 is eigen vector



Adding and scaling linear transformations

- adding / subtracting transformations

- Condition: Two linear transformation can be + or - only if both of them have same domain and codomain

$$\begin{array}{l} S: \mathbb{R}^n \rightarrow \mathbb{R}^m \\ T: \mathbb{R}^n \rightarrow \mathbb{R}^m \end{array}$$

- Addition is defined as

$$\begin{aligned} (S+T)(\vec{x}) &= S(\vec{x}) + T(\vec{x}) \\ &= A\vec{x} + B\vec{x} \\ &= (A+B)\vec{x} \end{aligned}$$

- Scaling transformations

$$cT(\vec{x})$$

c is scaling factor

$$c(B\vec{x})$$

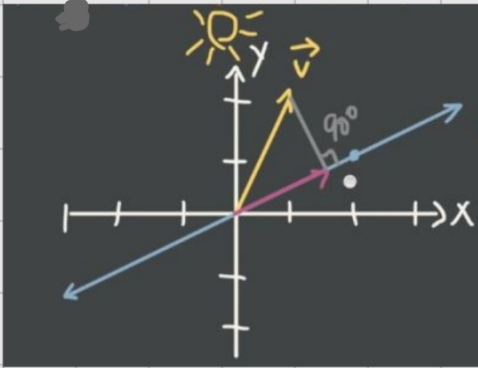
B is transformation matrix

$$(cB)\vec{x}$$

scaling factor gets multiplied with transformation matrix

Projection as linear transformation

- What is projection of a vector



- Finding projection of vector - method 1

$$\text{Proj}_L(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{x}}{\vec{x} \cdot \vec{x}} \right) \vec{x}$$

Here $\vec{x}$ is any positional vector along the line

- Finding projection of vector - method 2

$$\text{Proj}_L(\vec{v}) = (\vec{v} \cdot \hat{u}) \hat{u}$$

- Finding projection of vector - method 3 - projection as linear transformation

- Projection is linear transformation because it satisfies both condition for being linear transformation

- It is closed under addition

$$\text{Proj}_L(\vec{a} + \vec{b}) = ((\vec{a} + \vec{b}) \cdot \hat{u})\hat{u}$$

$$\text{Proj}_L(\vec{a} + \vec{b}) = (\vec{a} \cdot \hat{u} + \vec{b} \cdot \hat{u})\hat{u}$$

$$\text{Proj}_L(\vec{a} + \vec{b}) = (\vec{a} \cdot \hat{u})\hat{u} + (\vec{b} \cdot \hat{u})\hat{u}$$

$$\text{Proj}_L(\vec{a} + \vec{b}) = \text{Proj}_L(\vec{a}) + \text{Proj}_L(\vec{b})$$

- It is closed under scalar multiplication

$$\text{Proj}_L(c\vec{a}) = (c\vec{a} \cdot \hat{u})\hat{u}$$

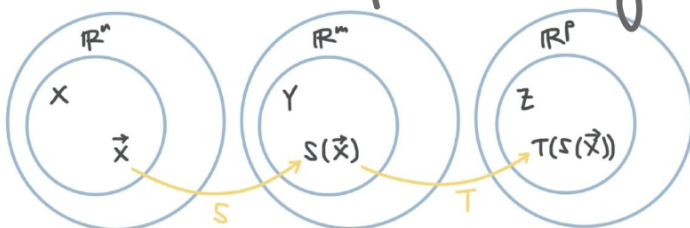
$$\text{Proj}_L(c\vec{a}) = c(\vec{a} \cdot \hat{u})\hat{u}$$

$$\text{Proj}_L(c\vec{a}) = c\text{Proj}_L(\vec{a})$$

- Hence projection can be defined using transformation matrix $\text{Proj}_L(\vec{v}) = A\vec{v}$. Once we find the transformation matrix (using how std vectors are transformed), it can be used to find projection of any other vector. This is the only method that even works in more than 3 dimensions

Composition of linear transformation

- What is composition of linear transformation



$T \circ S: X \rightarrow Z$ is composition of T and S

- composition of 2 linear transformation is itself a linear transformation because it satisfies both conditions for being linear transformation
 - closed under addition

$$T \circ S(\vec{x} + \vec{y}) = T(S(\vec{x} + \vec{y}))$$

$$T \circ S(\vec{x} + \vec{y}) = T(S(\vec{x}) + S(\vec{y}))$$

$$T \circ S(\vec{x} + \vec{y}) = T(S(\vec{x})) + T(S(\vec{y}))$$

$$T \circ S(\vec{x} + \vec{y}) = T \circ S(\vec{x}) + T \circ S(\vec{y})$$

- closed under scalar multiplication

$$T(S(c\vec{x})) = T(S(c\vec{x}))$$

$$T(S(c\vec{x})) = T(cS(\vec{x}))$$

$$T(S(c\vec{x})) = cT(S(\vec{x}))$$

- Because it is a linear transformation we can find its transformation matrix

$$T \circ S(\vec{x}) = T(S(\vec{x})) = T(A\vec{x}) = BA\vec{x} = C\vec{x}$$

- Example

$$S(\vec{a}) = \begin{bmatrix} 2 & 0 \\ -1 & 4 \end{bmatrix} \vec{a}$$

$$\vec{a} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$T(\vec{b}) = \begin{bmatrix} -1 & -2 \\ 3 & 0 \end{bmatrix} \vec{b}$$

Given

$$T(S(\vec{a})) = \begin{bmatrix} -1 & -2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & -8 \\ b & 0 \end{bmatrix}$$

$$C\vec{a} = \begin{bmatrix} 0 & -8 \\ b & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -16 \\ 18 \end{bmatrix}$$

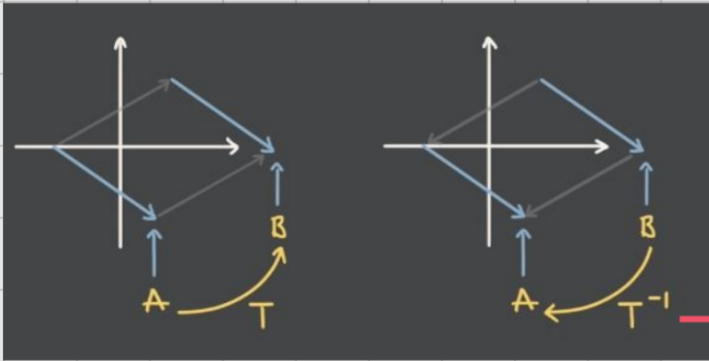
Transformation matrix of composition

Transformation of $\vec{a}$

Inverses

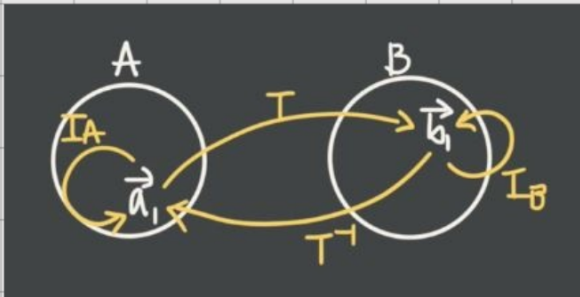
Inverse of Transformation

- Inverse of a transformation



undoes effect of transformation

- In other words inverse transformation maps back to original vector



$$\left. \begin{aligned} T^{-1}(T(\vec{a}_1)) &= I_A(\vec{a}_1) \\ T(T^{-1}(\vec{b}_1)) &= I_B(\vec{b}_1) \end{aligned} \right\}$$

- Conditions for inverse to exist - Inverse exist when the original transformation T is

T is surjective/onto:
every $\vec{b}$ is being mapped to
 T is injective/one-to-one:
only one $\vec{a}$ is mapping to a given $\vec{b}$

And if inverse exist these conditions are true

- Also T^{-1} is unique

Only square matrices are invertible

- Only square matrix can be invertible (However not all square matrix are invertible)

- If columns are more than rows

consider $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & -2 \end{bmatrix}$

its reduced row echelon form is $\begin{bmatrix} x_1 & x_2 & x_3 \\ 1 & 0 & 1/3 & | & 0 \\ 0 & 1 & -1 & | & 0 \end{bmatrix}$

its null space is $N(A) = \text{Span} \left(\begin{bmatrix} -1/3 \\ -1/3 \\ 1 \end{bmatrix} \right)$ many to one not injective
 $\begin{bmatrix} -1/3 \\ -1/3 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \vec{0}$

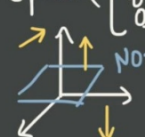
Hence when no of columns are more than no of rows the transformation matrix is not invertible

- If no of rows are more than no of columns

consider matrix $B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \\ 0 & -2 \end{bmatrix}$

Reduced row echelon form of B is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

column space of B is

$C(B) = \text{Span} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$
 $\mathbb{R}^3$  not surjective

span of 2 vectors cannot span 3D hence B is not onto or surjective hence transformation matrix cannot be invertible when rows are more than column

Matrix inverses, invertibles & singular matrices

Inverse of matrix: A matrix A is said to be invertible if there exist a matrix B such that $AB = BA = I$ and B is called inverse matrix of A and represented as A^{-1}

Find inverse using adjoint

$A^{-1} = \frac{\text{Adj } A}{|A|}$ $\rightarrow$ Transpose of cofactors
 $|A|$ $\rightarrow$ Determinant of A

Q. $A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$ find A^{-1}

Ans

Step 1: $|A| = 2(0-2) + 1(0-2)$
 $= -4 - 2$
 $= -6$

// $\neq 0$ hence A^{-1} exist

Step 2: find cofactor matrix ($C_{ij} = (-1)^{i+j} \times \text{minor}$)

$$= \begin{bmatrix} 1(0-2) & -1(0-2) & 1(0-1) \\ -1(0-0) & 1(0-0) & -1(2+1) \\ 1(-2-0) & -1(4-0) & 1(2-0) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2 & -1 \\ 0 & 0 & -3 \\ -2 & -4 & 2 \end{bmatrix}$$

Step 3: Transpose of cofactor matrix will give $\text{adj}(A)$

$$= \begin{bmatrix} -2 & 0 & -2 \\ 2 & 0 & -4 \\ -1 & -3 & 2 \end{bmatrix}$$

step 4:

$$A^{-1} = \frac{1}{-6} \begin{bmatrix} -2 & 0 & -2 \\ 2 & 0 & -4 \\ -1 & -3 & 2 \end{bmatrix}$$

Properties of inverse of matrix

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof:

$$(AB)(AB)^{-1} = I$$
$$A^{-1}AB(AB)^{-1} = A^{-1}I$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(A^T)^{-1} = (A^{-1})^T$$

Proof:

consider $(A^T)(A^{-1})^T$

// matrix multiplied with its inverse should give identity matrix

$$= (A^{-1}A)^T$$

$$// \quad BA = (AB)^T$$

$$= I$$

Properties of adjoint matrix

- $A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = I |A|$

// A is square matrix

Proof:

$$A^{-1} = \frac{\text{adj} A}{|A|}$$

$$AA^{-1} = \frac{A \cdot \text{adj} A}{|A|}$$

// premultiply by A

$$I |A| = A \text{adj} A$$

// Do post multiply by A to get second part

- $|\text{adj} A| = |A|^{n-1}$

Proof:

$$A \cdot \text{adj} A = |A| I$$

$$A \cdot \text{adj} A = \begin{bmatrix} |A| & 0 & 0 & - & - \\ 0 & |A| & 0 & - & . \\ \vdots & 0 & |A| & - & . \\ . & \vdots & . & & \end{bmatrix} \quad \begin{matrix} |A| \\ n \times n \end{matrix}$$

Take determinant on both side

$$|A \text{adj} A| = |A|^n$$

$$|A| |\text{adj} A| = |A|^n$$

$$|\text{adj} A| = |A|^{n-1}$$

• $\text{adj}(AB) = \text{adj}A \text{adj}B$ $\Rightarrow A \text{ \& B are not singular of same order}$
Proof:

$$(AB)\text{adj}(AB) = |AB|I \quad - (1)$$

* Consider $AB \cdot \underline{\text{adj}B} \cdot \text{adj}A$

$$A \cdot \underline{|B|} \cdot \text{adj}A$$

$$|B|A \text{adj}A$$

$$|B||A| = |AB| \quad - (2)$$

$$(1) = (2)$$

$$(AB)\text{adj}(AB) = AB \cdot \text{adj}B \cdot \text{adj}A$$

premultiply by $(AB)^{-1}$

$$(AB)^{-1}(AB)\text{adj}(AB) = (AB)^{-1}(AB) \cdot \text{adj}A \cdot \text{adj}B$$

$$\text{adj}(AB) = \text{adj}A \text{adj}B$$

✓ $(\text{adj } A)^T = \text{adj } A^T$

Proof:

We know

$$A \cdot \text{adj } A = |A| I_n$$

$$(A \cdot \text{adj } A)^T = (|A| I_n)^T$$

$$(\text{adj } A)^T A^T = |A| I_n \quad - (1)$$

also

$$\text{adj } A^T A^T = |A^T| I_n$$

$$\text{adj } A^T A^T = |A| I_n \quad - (2)$$

① = ② Hence proved

Theorem

Let $T: V_1 \rightarrow V_2$ be one one & onto linear transformation then $T^{-1}: V_2 \rightarrow V_1$ is linear & x

Proof: To show $T^{-1}(a\alpha + b\beta) = aT^{-1}(\alpha) + bT^{-1}(\beta)$

Let $\alpha, \beta \in V_1 \Rightarrow T(\alpha) = \alpha' \text{ \& } T(\beta) = \beta' \in V_2$
 $\Rightarrow \alpha = T^{-1}(\alpha') \text{ \& } \beta = T^{-1}(\beta')$ T is invertible

$$T \text{ is linear} \Rightarrow T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \\ = a\alpha' + b\beta'$$

$$a\alpha + b\beta = T^{-1}(a\alpha' + b\beta') \\ \checkmark aT^{-1}(\alpha') + bT^{-1}(\beta') = T^{-1}(a\alpha' + b\beta') \\ \text{Hence } T^{-1} \text{ is invertible}$$

Theorem

 Let $T: U \rightarrow V$ is linear & invertible $\Rightarrow TT^{-1} = T^{-1}T = I$

Proof: Let $\alpha \in U$ & $\beta \in V$ such that $T(\alpha) = \beta$

$$T^{-1}\beta = \alpha \\ TT^{-1}\beta = T\alpha = \beta \\ TT^{-1} = I$$

$$T\alpha = \beta \\ T^{-1}T\alpha = T^{-1}\beta \\ T^{-1}T\alpha = \alpha \\ T^{-1}T = I$$

Singular & non singular tx

$$T: U \rightarrow V$$

0 identity ele of U
0' identity ele of V

- T is singular tx if there exist $\alpha \neq 0 \in U$ s.t. $T(\alpha) = 0'$
- T is non singular if $N(T) = \{0\}$ i.e. $\alpha \in U$ s.t. $T(\alpha) = 0'$ then $\alpha = 0$

Theorem

$T: U \rightarrow V$ then T is non singular $\Leftrightarrow T$ is one one

Proof: let T be non singular & $\alpha, \beta \in U$
 $T(\alpha), T(\beta) \in V$

$$T(\alpha) = T(\beta)$$

$$T(\alpha) - T(\beta) = 0'$$

$$T(\alpha - \beta) = 0'$$

$$\alpha - \beta = 0$$

$$\alpha = \beta, \text{ i.e. } T \text{ is one one}$$

// T is linear tx

// T is non singular

II part, let T is one one

& given T is linear tx

let $T(\alpha) = 0'$ for some $\alpha \in U$

also $T(0) = 0'$

$$\alpha = 0$$

as T is one one

Determinants

Cramers Rule for solving systems

For example, given the linear system of two equations in two unknowns,

$$\underline{a_1}x + \underline{b_1}y = \underline{d_1}$$

$$\underline{a_2}x + \underline{b_2}y = \underline{d_2}$$

we can say that

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}, \text{ with } D \neq 0$$

where

$$D = \begin{vmatrix} \underline{a_1} & \underline{b_1} \\ \underline{a_2} & \underline{b_2} \end{vmatrix}, D_x = \begin{vmatrix} \underline{d_1} & \underline{b_1} \\ \underline{d_2} & \underline{b_2} \end{vmatrix}, D_y = \begin{vmatrix} \underline{a_1} & \underline{d_1} \\ \underline{a_2} & \underline{d_2} \end{vmatrix}$$

Or given a linear system of three equations in three unknowns,

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

we can say that

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}, \text{ with } D \neq 0$$

where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Properties of determinants

- multiplying by scalar

Scalar

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad |A|$$

$$kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} \quad |kA|$$

- Sum of rows

Sum of rows

$$\begin{bmatrix} \underline{2} & \underline{1} \\ \underline{3} & \underline{0} \end{bmatrix} + \begin{bmatrix} \underline{2} & \underline{1} \\ \underline{0} & \underline{4} \end{bmatrix} = \begin{bmatrix} \underline{2} & \underline{1} \\ \underline{3} & \underline{4} \end{bmatrix}$$

Only one row/column can be different that gets added or subtracted

- Swapping of rows

Swapped row

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix} \quad |A|$$

$$\begin{bmatrix} 2 & 2 \\ 3 & -1 \end{bmatrix} \quad -|A|$$

1 swap: -1
2 swaps: $(-1)(-1) = 1$
3 swaps: $(-1)^3 = -1$

// If rows are swapped more no of times

✓ Applying row operations do not change value of determinant (except scalar multiplication)

Row operations $R_1 - R_3 \rightarrow R_3$

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 2 & b \\ 3 & 1 & 0 \end{bmatrix} \quad |A|$$
$$|A| = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 2 & b \\ 0 & 0 & 0 \end{bmatrix} \quad |A|$$

Proof: Let $|A| = \begin{vmatrix} -r_1- \\ -r_2- \\ \vdots \\ -r_n- \end{vmatrix}$

then after row op $|A| = \begin{vmatrix} -r_1- \\ -r_2- \\ \vdots \\ -r_n - < r_2 - \end{vmatrix}$

also $|A| = \begin{vmatrix} -r_1- \\ -r_2- \\ \vdots \\ -r_n- \end{vmatrix} + \begin{vmatrix} -r_1- \\ -r_2- \\ \vdots \\ -<r_2- \end{vmatrix}$

$= 0$

- $|A| = |A^T|$

Prove by induction: show correct for 2×2
assume correct for $n \times n$
show correct for $(n+1) \times (n+1)$

- Determinant of skew symmetric matrix ($A = -A^T$) of odd order is 0

Proof:

We know $A^T = -A$

// A is skew symm

$$|A^T| = (-1)^n |A|$$

$$// |kA| = k^n |A|$$

n is order of A

$$|A| = (-1)^n |A| \quad // \quad |A^T| = |A|$$

$$(1 - (-1)^n) |A| = 0$$

$\neq 0$ when n is odd

then in that case $|A|$ have to be 0 for equality to hold

- $\det(A.B) = \det A \times \det B$

Proof:

case 1: $\det A = 0$

Then A is not invertable

AB is not invertable

$$\det(AB) = 0$$

$$\det(AB) = \det A \cdot \det B$$

case 2: $\det A \neq 0$

Then A is invertable

// A reduces to $rref$ by applying row ops that can be expressed as elementary transformation, conversely we can do elementary transformation on I to get A i.e. $A = E_1 E_2 \dots E_k I$, hence

$$\det(AB) = \det(E_1 E_2 \dots E_k B)$$

// multiplying by E is performing just row ops and does not chg determinant

$$\det(AB) = \det(E_1) \det(E_2 \dots E_k B)$$

$$= \det(E_1) \det(E_2) \dots \det(E_k) \det(B)$$

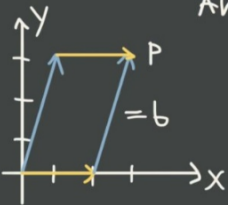
$$= \det(A) \det(B)$$

Determinant of triangle matrix

- use row operations to convert matrix into upper or lower triangular matrix as determinant of triangle matrix is just product of its diagonal entries

$$\begin{aligned}
 A &= \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & -1 & 0 \\ 1 & -3 & 2 & 2 \\ -1 & 0 & 1 & -2 \end{bmatrix} & \frac{1}{2}|A| &= \begin{vmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & 4 & -1 \end{vmatrix} & -\frac{1}{5}|A| &= \begin{vmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 0 & \frac{3}{5} \end{vmatrix} \\
 |A| &= \begin{vmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & -1 & 0 \\ 1 & -3 & 2 & 2 \\ -1 & 0 & 1 & -2 \end{vmatrix} & \frac{1}{2}|A| &= \begin{vmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{5}{2} & 1 \\ 0 & 0 & 4 & -1 \end{vmatrix} & -\frac{1}{5}|A| &= \frac{3}{5} \\
 |A| &= \begin{vmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & 4 & -1 \end{vmatrix} & \left(-\frac{2}{5}\right)\left(\frac{1}{2}\right)|A| &= \begin{vmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 4 & -1 \end{vmatrix} & |A| &= -3
 \end{aligned}$$

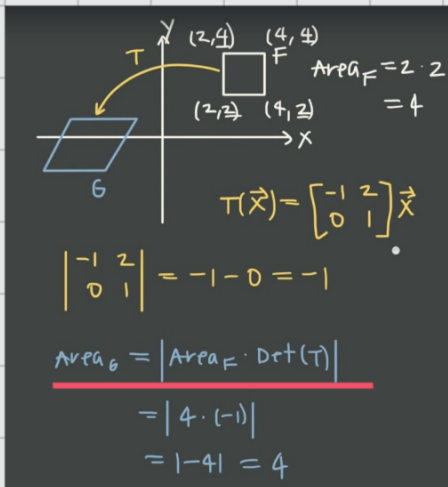
Determinants is area of ||gm

$$\begin{aligned}
 A &= \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \\
 \text{Area}_p &= |\det(A)| \\
 &= |0 - 6| \\
 &= |-6| \\
 &= 6
 \end{aligned}$$


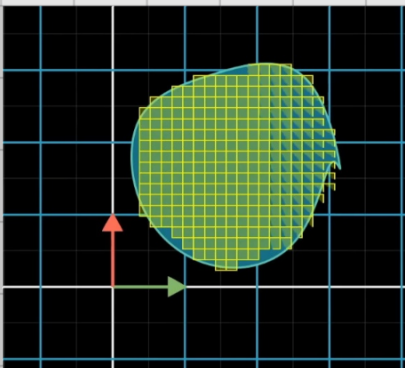
mod because area cannot be -ve

Geometric Interpretation of Determinant

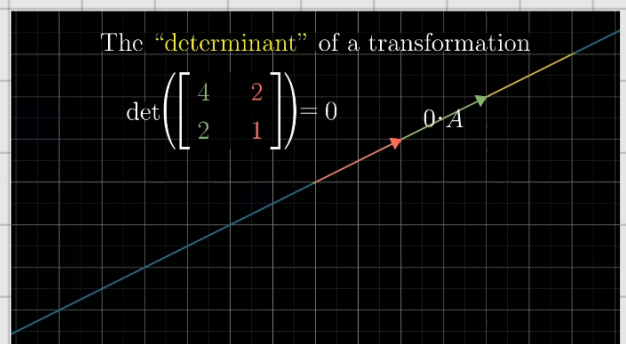
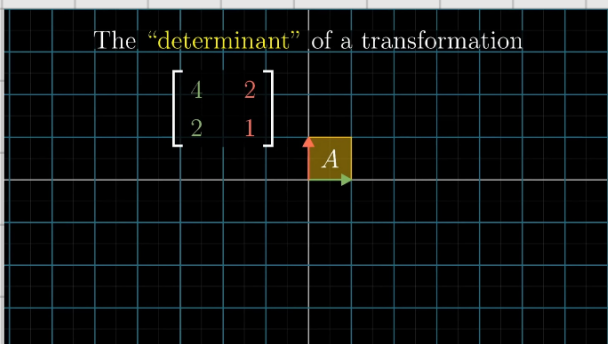
- Determinant of transformation matrix gives factor by which area changes after figure undergoes transformation



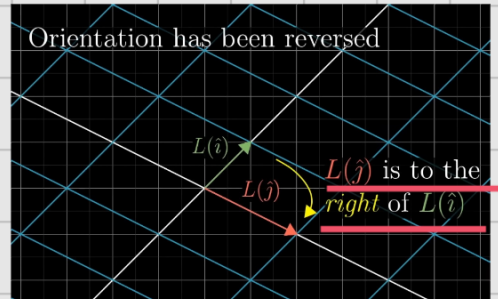
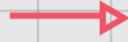
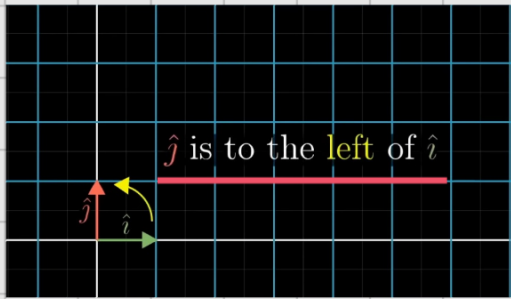
- # This is true even for uneven areas as they can also be thought to be made of very small squares



- # Determinant is 0 if it collapses space into smaller dimension

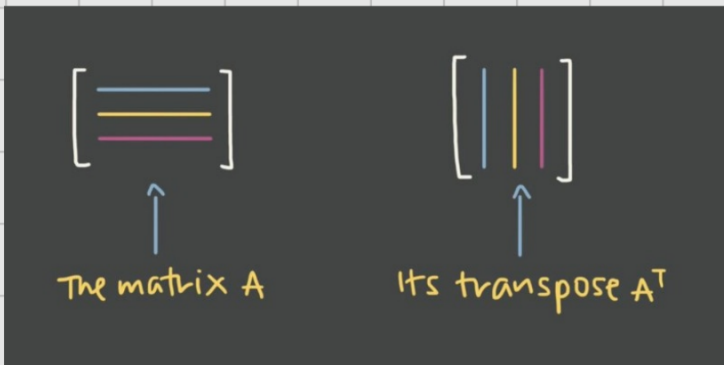


-ve determinant means that orientation of space has reversed



Transposes

Introduction



All matrices have transpose

Transposes and their determinants

- Determinant of matrix and its transpose are same if defined (Determinant only defined for square matrix)
- Example

$$\begin{aligned} |B| &= \begin{vmatrix} 3 & -4 & 1 \\ 0 & 0 & b \\ -1 & 2 & 2 \end{vmatrix} = -0 \begin{vmatrix} -4 & 1 \\ -2 & 2 \end{vmatrix} + 0 \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} - b \begin{vmatrix} 3 & -4 \\ -1 & 2 \end{vmatrix} \\ &= -b(b-4) = -b(2) = \underline{-12} \end{aligned}$$
$$\begin{aligned} |B^T| &= \begin{vmatrix} 3 & 0 & -1 \\ -4 & 0 & 2 \\ 1 & b & 2 \end{vmatrix} = -0 \begin{vmatrix} -4 & 2 \\ 1 & 2 \end{vmatrix} + 0 \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} - b \begin{vmatrix} 3 & -1 \\ -4 & 2 \end{vmatrix} \\ &= -b(b-4) = -b(2) = \underline{-12} \end{aligned}$$

Udemy

Transposes of products, sums and inverse

- Products

products

$$AB \quad (ABC)^T = C^T B^T A^T$$

$$(AB)^T = B^T A^T$$

order gets reversed

- sum (or difference)

sums

$$(A+B)^T = A^T + B^T$$

- Inverse

inverses

$$(A^T)^{-1} = (A^{-1})^T$$

- $A^T A$ is always a square matrix

$$A^T A = n \times n$$

$$(n \times m) \quad (m \times n)$$

It is invertable if column vectors of A are linearly independent

Properties of transpose

• Prove AA^T is symmetric

• $(kA)^T = kA^T$

Proof Take matrix kA means every ele of A multiply by k
then take transpose
then take out k as common

• $(AB)^T = B^T A^T$

Proof: $A = (a_{ij})$, $B = (b_{ij})$
 $A^T = (a_{ji})$, $B^T = (b_{ji})$

$$\text{then } (AB)^T = \sum_{k=1}^n (a_{ik} b_{kj})^T$$

$$= \sum_{k=1}^n a_{jk} b_{ki}$$

$$= \sum_{k=1}^n b_{ki} a_{jk}$$

$$= B^T A^T$$

Null and column space of transpose

$$\begin{aligned}A^T \vec{x} &= \vec{0} \\ (A^T \vec{x})^T &= (\vec{0})^T \\ \vec{x}^T A^T &= \vec{0}^T \\ \vec{x}^T A &= \vec{0}\end{aligned}$$

That is why it is called left null space of A

- Summary :

Subspace	Symbol	Space	Dimension
Column space of A	$C(A)$	$\mathbb{R}^m$	$\text{Dim}(C(A)) = r$
Null space of A	$N(A)$	$\mathbb{R}^n$	$\text{Dim}(N(A)) = n - r$
Row space of A (column space of A^T)	$C(A^T)$	$\mathbb{R}^n$	$\text{Dim}(C(A^T)) = r$
Left null space of A (null space of A^T)	$N(A^T)$	$\mathbb{R}^m$	$\text{Dim}(N(A^T)) = m - r$

Rank of A and A transpose are same

or $\text{rank of } A = \text{rank of } A^T$
or $\dim \text{ of column space} = \dim \text{ of row space.}$

Proof: $A = \left[\begin{array}{c|c|c|c|c} \text{red} & \text{green} & \text{yellow} & \text{blue} & \dots \\ r_1 & r_2 & r_3 & r_4 & \dots \\ \hline & & & & \end{array} \right]$

$\downarrow \text{rref}$

$$A = \left[\begin{array}{cccc|cccc} 1 & 0 & x & 0 & - & - & - & - \\ 0 & 1 & x & 0 & - & - & - & - \\ 0 & 0 & 0 & 1 & - & - & - & - \\ \vdots & \vdots & & & & & & \\ 0 & 0 & 0 & 0 & - & - & - & 0 \end{array} \right]$$

this row will be part of basis of row space as no other row has 1 at first place

Hence pivot rows in $\text{rref}(A)$ are basis for row space or column space of A

and no of pivot row = no of pivot entries
that also equal no of pivot column

Properties of rank of matrix

- Matrix is non singular $\Leftrightarrow$ matrix is full rank
- Rank of $m \times n$ matrix can be atmost smaller of m or n
- Rank of 0 matrix is defined as 0
- Rank of identity matrix of order n is n
- Rank of $AB \leq \min(\text{rank of } A, \text{rank of } B)$
- Rank of $A+B \leq \text{rank } A + \text{rank } B$
- Rank $AA^T = \text{Rank of } A$
- Rank of $A = \text{Rank of } A^T$
- Rank of $AB \neq \text{Rank of } BA$

Orthogonality & change of basis

Intro

$$V = \text{Span} \left(\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} \right)$$

1. Direction
~~2. Length~~

Pick better vectors

$$V = \text{Span} \left(\begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix} \right) \leftarrow \text{change of basis}$$

Orthogonal complement

- two subspaces are said to be orthogonal to each other when every vector of first subspace is $\perp$ to every vector of second subspace. Then the subspaces are orthogonal complement of one another

$$\mathbb{R}^n \quad V = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_n \}$$
$$\mathbb{R}^n \quad V^\perp = \{ \vec{x}_1, \vec{x}_2, \vec{x}_3, \dots, \vec{x}_m \}$$
$$V^\perp = \{ \vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{v} = 0 \text{ for every } \vec{v} \in V \}$$

- orthogonal complement of subspace is also a subspace as it is closed under addition and multiplication

$$\begin{array}{ll} \vec{x}_1 \cdot \vec{v} = 0 & \vec{x}_1 \cdot \vec{v} + \vec{x}_2 \cdot \vec{v} = 0 \\ \vec{x}_2 \cdot \vec{v} = 0 & (\vec{x}_1 + \vec{x}_2) \cdot \vec{v} = 0 \end{array}$$

$$c\vec{x}_1 \cdot \vec{v} = 0 \quad c(\vec{x}_1 \cdot \vec{v}) = 0 \quad c(0) = 0$$
$$0 = 0$$

- Example - find orthogonal subspace of

$$V = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right)$$

Ans $V^\perp = \{ \vec{x} \in \mathbb{R}^3 \mid \vec{x} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 0 \text{ and } \vec{x} \cdot \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = 0 \}$

Let $\vec{x} = (x_1, x_2, x_3)$

$$\begin{aligned} 1x_1 + 0x_2 + 2x_3 &= 0 \\ -1x_1 + 1x_2 + 3x_3 &= 0 \end{aligned}$$

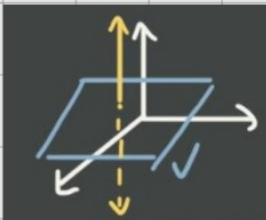
Dot product of $\vec{x}$ with the two vectors is 0

$$\begin{bmatrix} 1 & 0 & 2 & | & 0 \\ -1 & 1 & 3 & | & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & 5 & | & 0 \end{bmatrix}$$
$$\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 + 5x_3 &= 0 \end{aligned}$$

Solve eqns

$$\begin{aligned} x_1 &= -2x_3 \\ x_2 &= -5x_3 \end{aligned}$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix}$$

$$V^\perp = \text{span} \left(\begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix} \right)$$

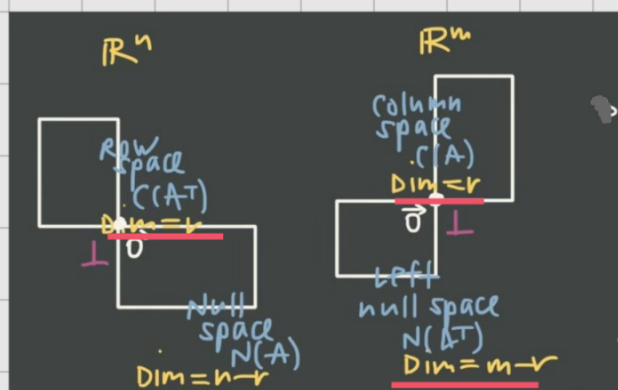


One would have thought orthogonal complement of plane would be a plane $\perp$ to the plane, however vectors at intersection of both plane are common to both and violate property of orthogonal complement

$$(V^\perp)^\perp = V$$

Orthogonal complements of fundamental subspaces

- 4 fundamental subspaces of a matrix



Dimension of subspace and its orthogonal complement is equal to dimension of space containing them

$$\begin{aligned} N(A) &= (C(A^T))^{\perp} & C(A) &= (N(A^T))^{\perp} \\ (N(A))^{\perp} &= C(A^T) & N(A^T) &= (C(A))^{\perp} \end{aligned}$$

- check relation for matrix A

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 0 & 2 \\ -1 & 1 & 2 & 0 \\ 2 & 0 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \end{aligned}$$

space		Dim
$C(A)$	$\mathbb{R}^3$	3
$N(A)$	$\mathbb{R}^4$	1
$C(A^T)$	$\mathbb{R}^4$	3
$N(A^T)$	$\mathbb{R}^3$	0

$$\begin{aligned} C(A) &= \text{span} \left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right) \\ N(A) &= \text{span} \left(\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} \right) \\ C(A^T) &= \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right) \\ N(A^T) &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

Projection onto subspace

- In gen we can project the vector ($\vec{x}$) on any subspace using the formula

$v \text{ in } \mathbb{R}^n$

$$\text{proj}_v \vec{x} = \underline{A(A^T A)^{-1} A^T \vec{x}}$$

Here A is matrix of column vectors that form basis for subspace

Just the transformation matrix

Least square solution

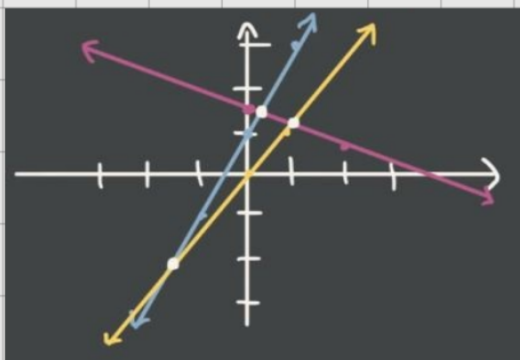
- $A\vec{x} = \vec{b}$ does not have a soln if $\vec{b}$ not in column space of A
- We can find the closest approximation of soln by projecting $\vec{b}$ to column space of A and solving for $\vec{x}$
- This soln is represented as $\vec{x}^*$ and obtained using the formula

$$A^T A \vec{x}^* = A^T \vec{b}$$

- Example - solve set of eqns

$$\begin{cases} -2x+y=1 \\ -x+y=0 \\ x+2y=3 \end{cases}$$

Ans clearly the 3 lines do not intersect at 1 pt
hence there is no soln



Hence we can find least square soln

$$\underbrace{\begin{bmatrix} b & -1 \\ -1 & b \end{bmatrix}}_{A^T A} \underbrace{\vec{x}^*}_{A^T} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{A^T}$$

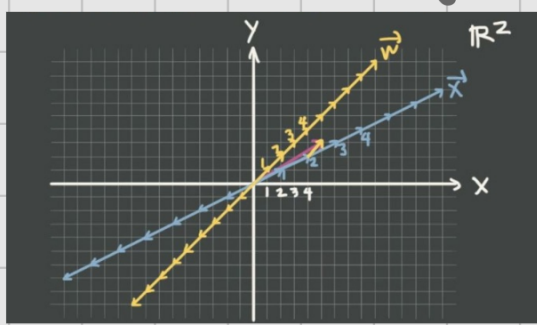
$\vec{x}^* = \begin{bmatrix} 13/35 \\ 43/35 \end{bmatrix}$ This pt has min $\sum (\text{dist of vertices from pt})^2$
ie best approximate soln

Coordinates in a new basis

- Imagine we want to start using $\begin{cases} \vec{x} = (2, 1) \\ \vec{w} = (1, 1) \end{cases}$ as basis

vector instead of std basis vector

$$\begin{aligned} i &= (1, 0) \\ j &= (0, 1) \end{aligned}$$



How to find $\vec{v}_B$, i.e. $\vec{v}$ in terms of new basis vectors

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} [\vec{v}]_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

change of basis matrix
 $\vec{v}$ in terms of new basis vectors
vector $\vec{v}$ is expressed in std basis

$$[\vec{v}]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

we can also find how will a vector be represented in new basis

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4+1 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

If A is invertible we can find a gen eqn

$$A[\vec{v}]_B = \vec{v}$$

$$A^{-1}A[\vec{v}]_B = A^{-1}\vec{v}$$

$$I[\vec{v}]_B = A^{-1}\vec{v}$$

$$[\vec{v}]_B = A^{-1}\vec{v}$$

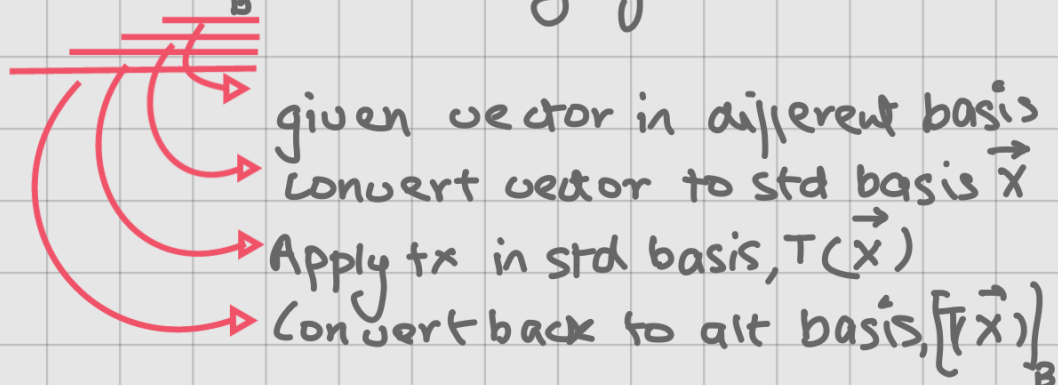
Transformation Matrix for basis

- Transforming from alternate basis

Till now we were transforming in std basis ie
 $T(\vec{x}) = A\vec{x}$

given the transformation in std basis we can perform the same transformation in terms of different basis

$$[T(\vec{x})]_B = C^{-1} A C [\vec{x}]_B \quad // C \text{ is chg of basis matrix}$$



- Example

Given

$$T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix} \vec{x}$$

$$B = \text{span}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix}\right)$$

$$[\vec{x}]_B = (1, 3)$$

Apply transformation in different basis

Ans

$$C^{-1} A C^{-1} = \begin{bmatrix} -1 & -\frac{2}{3} \\ 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} & -\frac{26}{3} \\ \frac{2}{3} & -\frac{10}{3} \end{bmatrix}$$

$$\text{Transformed matrix} = \frac{1}{3} \begin{bmatrix} 7 & -26 \\ 2 & -10 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

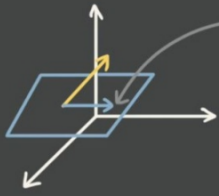
Orthonormal bases and Gram-Smith

Intro

Orthonormal set:

1. All vectors are normal
2. All vectors are orthogonal

length = 1



Easier onto an orthonormal subspace

Orthonormal basis

- orthonormal basis - set of vectors in which every vector is orthogonal to every other vector in set and every vector's length is 1

Orthonormal bases

orthogonal $\vec{v}_1 \cdot \vec{v}_2 = 0$

Normal $\vec{v}_1 \cdot \vec{v}_1 = \|\vec{v}_1\|^2 = 1$

- Example - check if given set of vector form orthonormal basis

$$\vec{v}_1 = (0, 0, 1)$$

$$\vec{v}_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\vec{v}_3 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$$

Ans They are orthogonal

$$\vec{v}_1 \cdot \vec{v}_2 = 0 + 0 + 0 = 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = 0 + 0 + 0 = 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = \frac{1}{2} + \left(-\frac{1}{2}\right) + 0 = 0$$

They are normal

$$\|\vec{v}_1\|^2 = 0 + 0 + 1 = 1$$

$$\|\vec{v}_2\|^2 = \frac{1}{2} + \frac{1}{2} + 0 = 1$$

$$\|\vec{v}_3\|^2 = \frac{1}{2} + \frac{1}{2} + 0 = 1$$

Hence the given set of vectors form orthonormal Basis

Projection onto orthonormal basis

- formula for projection simplified down drastically when we are projecting on subspace spanned by orthonormal basis

$$\begin{aligned}\text{proj}_V \vec{x} &= A(A^T A)^{-1} A^T \vec{x} \\ &= A I A^T \vec{x} \\ &= \underline{A A^T} \vec{x}\end{aligned}$$

$A^T A = I$ for orthonormal matrix

 Projection matrix

- Example - Find projection matrix for given orthonormal basis

$$\vec{v}_1 = (0, 0, 1)$$
$$\vec{v}_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

Ans

$$A = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 + \frac{1}{2} & 0 + \frac{1}{2} & 0 + 0 \\ 0 + \frac{1}{2} & 0 + \frac{1}{2} & 0 + 0 \\ 0 + 0 & 0 + 0 & 1 + 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}$$

Gram - Schmidt process for change of basis

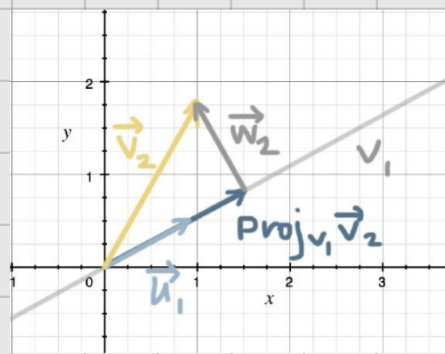
- The process defines steps to convert a set of vectors spanning a subspace to orthonormal basis spanning same subspace

- step 1: Given $V = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$

- step 2: Normalize first vector: $\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$
 & update $V = \text{span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$

- step 3: Find $\vec{w}_2$ $\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1$

// w_2 is



// w_2 is in same plane as $\vec{v}_2$ but normal to $\vec{v}_1$

- step 4: Normalize $\vec{w}_2$ and update $V = \text{Span}(\vec{u}_1, \vec{u}_2, \vec{v}_3)$

- step 5: keep making other vectors normal to previous vectors, normalize them and update V

$$\vec{w}_3 = \vec{v}_3 - [(\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 + (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2]$$

...

$$\vec{w}_4 = \vec{v}_4 - [(\vec{v}_4 \cdot \vec{u}_1)\vec{u}_1 + (\vec{v}_4 \cdot \vec{u}_2)\vec{u}_2 + (\vec{v}_4 \cdot \vec{u}_3)\vec{u}_3]$$

Eigen values

&

Eigen vectors

Eigen value & eigen vector

- Let $T: V \rightarrow V$ is a linear tx on n dim vector space over field F , then scalar $\lambda \in F$ is called eigen value of T if there exist a non zero vector $\alpha \in V$ such that

$$T(\alpha) = \lambda \alpha$$

Here α is called eigen vector corresponding to eigen value λ

Intro

$$T(\vec{v}) = \lambda \vec{v}$$

↑ Eigenvalue

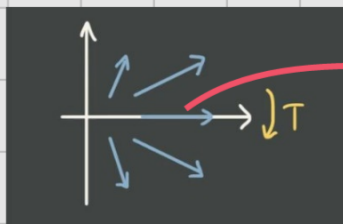
← Eigenvector

↓ Forms an Eigenbasis

Eigen values, eigen Vectors and eigen spaces

- consider a linear transformation $T(\vec{v}) = A\vec{v}$

Eigen vector are those vector that do not change their axis even after transformation $A\vec{v} = \lambda \vec{v}$ however their magnitude may change by factor of λ (Eigen value)



Eigen vector for transformation T

- How to find eigen value and eigen vector

$$A\vec{v} = \lambda\vec{v}$$

$$\vec{0} = \lambda\vec{v} - A\vec{v}$$

$$\vec{0} = \lambda \underline{I_n} \vec{v} - A\vec{v}$$

$$\vec{0} = \underbrace{(\lambda I_n - A)}_B \vec{v}$$

$$B\vec{v} = \vec{0}$$

$$|B| = 0$$

$$|\lambda I_n - A| = 0$$

Defination of eigen vector (transformation does not chg it dir, just chg length)

multiply by I_n to make it matrix x vector

v is in null space of matrix B

If v exists then null space of B has more than just $\vec{0}$

That means B has some linear dependent column vectors

Hence determinant of $B = 0$

largest eigen value called spectral radius of A

sum of eigen values = trace of transformation matrix
ie sum of diagonal

$$3 + (-2) = \text{Trace of } \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$$

$$1 = 1$$

Product of eigen values = determinant of transformation matrix

$$3 \times (-2) = \text{determinant of } \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$$

$$-6 = -2 - (4)$$

$$-6 = -6$$

- Finding eigen space corresponding to eigen value of $\lambda = -2$

$$E_{\lambda} = N(\lambda I_n - A)$$

$$E_{-2} = N\left(\begin{bmatrix} -4 & 4 \\ 1 & -1 \end{bmatrix}\right)$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} v_1 - v_2 = 0 \\ v_1 = v_2 \\ v_2 = t \end{array}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$E_{-2} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

→ Eigen space is null space of B

- similarly for $\lambda = 3$ ie

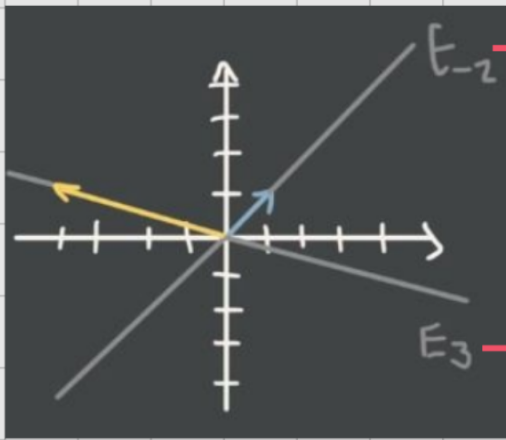
$$E_3 = N\left(\begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix}\right)$$

$$\left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} v_1 + 4v_2 = 0 \\ v_1 = -4v_2 \\ v_2 = t \end{array}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$E_3 = \text{span}\left(\begin{bmatrix} -4 \\ 1 \end{bmatrix}\right)$$

- Geometrical meaning of eigen value and eigen space



→ All vectors in E_{-2} subspace after undergoing transformation will be flipped along same axis and scaled by 2

→ All vectors in E_3 subspace after transformation will point in same direction and will get scaled by factor of 3

Eigen in 3D

- Process for finding eigen value and eigen space is same in 3D

Example find eigen value and eigen space for

$$T(\vec{v}) = \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix} \vec{v}$$

Ans

$$|\lambda I_n - A| = \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix} \right|$$

$$\lambda = -6, -1, 5$$

$$E_{-6} = \text{span} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

$$E_{-1} = \text{span} \left(\begin{bmatrix} -\frac{1}{5} \\ -\frac{1}{25} \\ 1 \end{bmatrix} \right)$$

$$E_5 = \text{span} \left(\begin{bmatrix} 0 \\ -\frac{3}{11} \\ 1 \end{bmatrix} \right)$$

- When eigen values are same

$$Q, A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Ans $\lambda = 2, 2, 8$
For $\lambda = 2$ we get

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_1 - x_2 + x_3 = 0 \quad // \text{ 3 variable 1 eqn}$$

$$\text{let } x_3 = 1, x_2 = -1 \Rightarrow x_1 = -1 \quad // \text{ choose any value}$$

$$\text{eigen vector} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{let } x_3 = 0, x_2 = 2 \Rightarrow x_1 = 1$$

$$\text{eigen vector} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

We could have chosen any other value of x_2, x_3 . The eigen space for λ is actually a plane spanned by 2 vectors

Theorem 1

If α is an eigen vector for λ eigen value then $k\alpha$ is also eigen vector for λ

Proof:

$$T(\alpha) = \lambda\alpha$$

$$\text{then } T(k\alpha) = kT(\alpha)$$

// T is linear $\neq x$

$$= k(\lambda\alpha)$$

$$= (k\lambda)\alpha$$

$$= (\lambda k)\alpha$$

$$= \lambda(k\alpha)$$

Hence $k\alpha$ is eigen vector

Theorem 2

If α is eigen vector of T then α cannot correspond to more than one eigen value of T

Proof by contradiction:

Let α be eigen vector of T for $\lambda_1 \neq \lambda_2$ we will show $\lambda_1 = \lambda_2$

$$T(\alpha) = \lambda_1\alpha$$

$$T(\alpha) = \lambda_2\alpha$$

$$T(\alpha) = T(\alpha)$$

T is one one

$$\lambda_1\alpha = \lambda_2\alpha$$

$$(\lambda_1 - \lambda_2)\alpha = 0$$

$$\lambda_1 = \lambda_2$$

as $\alpha \neq 0$

Theorem 3

Eigen space of any eigen value λ of T is subspace of vector space

Proof: Prove property of linear subspace

Theorem 4

Distinct non zero eigen vector of T corresponding to distinct eigen value are linearly independent

Proof: Given $T(x_i) = \lambda_i x_i$

To prove $S = \{x_1, x_2, \dots, x_n\}$ are LI
we will prove by induction

for $n=1$, S contain only 1 vector that is LI

for $n=k$, ie $S = \{x_1, x_2, \dots, x_k\}$ is LI

Show for $n=k+1$, $S = \{x_1, x_2, \dots, x_{k+1}\}$ is LI

ie ① $a_1 x_1 + a_2 x_2 + \dots + a_{k+1} x_{k+1} = 0$ then all $a = 0$

take transformation on both side

$$T(a_1 x_1 + \dots + a_{k+1} x_{k+1}) = T(0) = 0$$

$$\textcircled{2} a_1 \lambda_1 x_1 + a_2 \lambda_2 x_2 + \dots + a_{k+1} \lambda_{k+1} x_{k+1} = 0$$

$$\textcircled{2} - (\lambda_{k+1} \textcircled{1})$$

$$a_1(\lambda_1 - \lambda_{k+1})\alpha_1 + a_2(\lambda_2 - \lambda_{k+1})\alpha_2 - \dots - a_k(\lambda_k - \lambda_{k+1})\alpha_k + 0 = 0$$

since $\alpha_1, \alpha_2, \dots, \alpha_k$ are LI and $\lambda_i \neq \lambda_{k+1}$ distinct
 hence $a_1, a_2, \dots, a_k = 0$ put in ①
 we get $a_{k+1} = 0$, hence proved

Theorem 5

n dim vector space cannot have more than
 n distinct eigen values

Proof by contradiction: Let there be $n+1$ eigen value
 then there will be $n+1$ LI eigen vectors
 but n dimension cannot have $n+1$ LI vectors

Properties of eigen values

✓ A & A^T have same eigen values

Proof:

$$\text{consider } (A - \lambda I)^T \\ = A^T - \lambda I^T$$

$$\Rightarrow (A - \lambda I)^T = A^T - \lambda I$$

take determinant

$$|(A - \lambda I)^T| = |A^T - \lambda I|$$

$$|A - \lambda I| = |A^T - \lambda I| \quad \text{'' } |A| = |A^T|$$

A & A^T have same char eqn hence eigen values

• Eigen value of Δ matrix are just diagonal elements of matrix.

Proof:

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & 0 & 0 \\ a_{21} & a_{22} - \lambda & 0 \\ a_{31} & a_{32} & a_{33} - \lambda \end{bmatrix} = 0$$

$$(a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) = 0$$

eigen value of identity matrix are all 1

✓ Sum of eigen value of matrix is sum of element of principal diagonal

Proof: Let the λ value of 3×3 A matrix be $\lambda_1, \lambda_2, \lambda_3$

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{bmatrix} = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

expand determinant way + take coeff of λ^2 expand by multiply & get coeff of λ^2

$$a_{11} + a_{22} + a_{33} = \lambda_1 + \lambda_2 + \lambda_3$$

✓ Product of eigen value of a matrix is equal to its determinant

Proof: Let the λ value of 3×3 A matrix be $\lambda_1, \lambda_2, \lambda_3$

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{bmatrix} = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

expand along row 1 & compare constant terms
 $|A|$

expand & compare const terms
 $\lambda_1 \lambda_2 \lambda_3$

✓ If $|A| = 0$ then atleast 1 eigen value will be 0
+ If $|A| \neq 0$ then no eigen value can be 0

If λ is eigen value of A then, $\frac{1}{\lambda}$ ($\lambda \neq 0$) is

eigen value of A^{-1}

Proof:

Let $\vec{x}$ be eigen vector corresponding to λ
then

$$A\vec{x} = \lambda\vec{x}$$

Premultiply by A^{-1} on both side

$$A^{-1}A\vec{x} = A^{-1}\lambda\vec{x}$$

$$\vec{x} = A^{-1}\lambda\vec{x}$$

$$\frac{1}{\lambda}\vec{x} = A^{-1}\vec{x}$$

$$\text{ie } A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$$

If λ is eigen value of A then $\frac{1}{\lambda}$ is eigen value of A^{-1}

If λ is eigen value of orthogonal matrix A
then $\frac{1}{\lambda}$ is also its eigen value

Proof we know $\frac{1}{\lambda}$ is eigen value of A^{-1} that is
equal to A

• If $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigen value of matrix A , then A^m has eigen values $\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$

Proof:

Let $\vec{x}_i$ be eigen vector corresponding to λ_i

then $A\vec{x}_i = \lambda_i\vec{x}_i$

consider

$$\begin{aligned} A^2\vec{x}_i &= A(A\vec{x}_i) \\ &= A(\lambda_i\vec{x}_i) \\ &= \lambda_i(A\vec{x}_i) \\ &= \lambda_i(\lambda_i\vec{x}_i) \\ &= \lambda_i^2\vec{x}_i \end{aligned}$$

ie $A^2\vec{x}_i = \lambda_i^2\vec{x}_i$

Similarly $A^m\vec{x}_i = \lambda_i^m\vec{x}_i$

ie λ_i^m is eigen value for A^m

• Eigen value of idempotent matrix ($A^2=A$) are 0 or 1

• Eigen value of involuntary matrix ($A=A^{-1}$) are -1 or 1
($A^2=I$)

• Eigen value of kA are $k\lambda_1, k\lambda_2, \dots, k\lambda_n$

Proof:

$$Ax = \lambda x$$

$$kAx = k\lambda x$$

$$(kA)x = (k\lambda)x \quad \text{Hence proved}$$

• Eigen value of $A - kI$ are $\lambda_1 - k, \lambda_2 - k, \dots$

✓ If $|A| \neq 0$ and λ is eigen value of A then
eigen value of $\text{adj } A$ is $\frac{|A|}{\lambda}$
Proof

$$AX = \lambda X$$
$$(\text{adj } A) AX = (\text{adj } A) \lambda X$$

$$|A| I X = \lambda (\text{adj } A) X$$

$$\left(\frac{|A|}{\lambda} \right) X = (\text{adj } A) X$$

is eigen value of $\text{adj } A$

$$AA^{-1} = A (\text{adj } A) = I$$

$$\Rightarrow A (\text{adj } A) = |A| I$$

• Eigen value of similar matrix is same

Proof To prove eigen value of A & $P^{-1}AP$ are same

consider $P^{-1}AP - \lambda I$

$$P^{-1}AP - \lambda P^{-1}P$$
$$P^{-1}(A - \lambda I)P$$

$$\Rightarrow \det(P^{-1}AP - \lambda I) = \det(P^{-1}(A - \lambda I)P)$$
$$= \det(P^{-1}) \det(A - \lambda I) \det(P)$$
$$= 1 \det(A - \lambda I) \det(P)$$

$$\Rightarrow \det(P^{-1}AP - \lambda I) = \det(A - \lambda I) = 0$$

Hence λ is also eigen value of $P^{-1}AP$

≠ similar matrix have same eigen value but
eigen vectors corresponding to eigen value will
be different.

* Hermitian matrix

matrix satisfy condition $A^\theta = (\bar{A})^T = A$
ie conjugate of $A = A$ transpose
eg $A = \begin{bmatrix} 4 & 1-i & 7 \\ 1+i & 6 & -i \\ 7 & i & 7 \end{bmatrix}$

* skew hermitian matrix

A is said to be skew hermitian matrix if
 $(\bar{A})^T = -A$

* Every diagonal element of skew hermitian matrix is either 0 or purely imaginary

* Unitary matrix: Matrix A is unitary matrix if

$$A A^\theta = I \quad \text{ie} \quad A(\bar{A})^T = I$$

All orthogonal matrices are unitary matrices

✓ Eigen value of hermitian matrix are always real

Proof: Let λ be eigen value & $\vec{x}$ be corresponding eigen vector

$$A\vec{x} = \lambda\vec{x}$$

$$x^0 A x = x^0 (\lambda x) = \lambda x^0 x \quad - (1)$$

Take conjugate transpose on both side

$$\Rightarrow (x^0 A x)^0 = (\lambda x^0 x)^0$$

$$\Rightarrow x^0 A (x^0)^0 = \bar{\lambda} x^0 (x^0)^0$$

$$\Rightarrow x^0 A x = \bar{\lambda} x^0 x \quad - (2)$$

from (1) & (2)

$$\lambda x^0 x = \bar{\lambda} x^0 x$$

$$(\lambda - \bar{\lambda}) x^0 x = 0$$

$$\lambda - \bar{\lambda} = 0 \quad \text{as } x \text{ is non 0}$$

$$\lambda = \bar{\lambda} \quad \lambda \text{ is real}$$

✓ Eigen value of symm matrix are always real

✓ Eigen value of skew hermitian matrix are purely imaginary or 0

Proof: let λ be eigen value of eigen vector x of skew hermitian matrix A

$$\begin{aligned} Ax &= \lambda x \\ i(Ax) &= i(\lambda x) \\ (iA)x &= (i\lambda)x \end{aligned}$$

$i\lambda$ is eigen value of iA
 iA is hermitian matrix
hermitian matrix has real λ
 $\Rightarrow i\lambda$ is real $\Rightarrow \lambda$ is purely imaginary

If A is hermitian
then iA is skew her...

$$\begin{aligned} \text{Proof: } (iA)^0 &= \bar{i}A^0 \\ &= -i(-A) \\ &= iA \end{aligned}$$

& if A is skew her...
then iA is hermi...

✓ Eigen value of skew symm matrix are purely imaginary or 0

✓ Prove - eigen value of unitary matrix are of unit modulus

Proof: Let λ be eigen value of eigen vector x of unitary matrix A

$$Ax = \lambda x \quad - (1)$$

$$(Ax)^0 = (\lambda x)^0$$

$$x^0 A^0 = \bar{\lambda} x^0 \quad - (2)$$

multiply (1) x (2) we get

$$(x^0 A^0)(Ax) = (\bar{\lambda} x^0)(\lambda x)$$

$$x^0 x = |\lambda|^2 x^0 x$$

$$(1 - |\lambda|^2) x^0 x = 0$$

$$|\lambda| = 1$$

hence proved

• Eigen value of A^0 are conjugate of eigen value of A

✓ Fun fact eigen value of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0, 2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 0, 0, 3$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = 0, 0, 0, 4$$

✓ Eigen vector of symm matrix are orthogonal

Proof: $Ax = \lambda x$ & $Ay = \gamma y$ be 2 eigen vector

$$\Rightarrow (Ax)^T = (\lambda x)^T$$

$$x^T A^T = \lambda x^T$$

$$\underline{x^T A^T y = \lambda x^T y}$$

$$x^T A y = x^T \gamma y$$

$$\underline{x^T A^T y = x^T \gamma y}$$

$$\lambda x^T y = x^T \gamma y$$

$x^T y$ is 0 as $\lambda \neq \gamma$

• Rank of matrix = no of non zero eigen values

✓ AB and BA have same eigen values

Proof: Let λ be eigen value of AB

$$ABx = \lambda x$$

$$BABx = B\lambda x$$

$$(BA)Bx = (\lambda)Bx$$

λ was eigen value of AB and is eigen value of BA, similarly show all eigen value of BA are also eigen value of AB hence AB and BA have same eigen values.

Although eigen vector may be different

• AA^T and $A^T A$ have same eigen values

Although if A is $n \times m$ matrix
then AA^T is $n \times n$ matrix
and $A^T A$ is $m \times m$ matrix

Let λ be eigen value of $A^T A$ then $A^T A x = \lambda x$

$$\Rightarrow A A^T A x = A \lambda x$$
$$(A A^T) A x = (\lambda) A x$$

$\Rightarrow \lambda$ was eigen value of $A^T A$ and is also eigen value of AA^T

Although eigen vector may be different

• Eigen value of symm matrix corresponding to distinct eigen value are orthogonal

Proof let x & y be 2 LI eigen vector corresponding to λ_1 & λ_2

Hence $Ax = \lambda_1 x$ & $Ay = \lambda_2 y$

take transpose
both side

$$y^T A x = \lambda_1 y^T x$$

$$x^T A^T y = \lambda_1 x^T y$$

$$\underline{x^T A y} = \lambda_1 x^T y$$

$$\underline{x^T A y} = \lambda_2 x^T y$$

$$\lambda_1 x^T y = \lambda_2 x^T y$$

$$(\lambda_1 - \lambda_2) x^T y = 0$$

$$\lambda_1 \neq \lambda_2 \text{ hence } x^T y = 0$$

$$x \perp y$$

Carley Hamilton theorem

Every square matrix satisfies its characteristic eqn
not necessary invertible can be singular or non singular
This theorem is used to find A^{-1}

Find inverse

Q consider $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

Ans char eqn is $\lambda^2 - 4\lambda - 5$

By Carley Hamilton theorem $A^2 - 4A - 5I = 0$

ie $A^{-1}(A^2 - 4A - 5I) = 0$

$$A - 4I - 5A^{-1} = 0$$

Now find $A^{-1} = \frac{1}{5}(A - 4I)$

Find higher powers

Q consider $F = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Ans Then char eqn $F^3 = 4F^2 - 3F$

multiply by F we get $F^4 = 4F^3 - 3F^2$
 $= 4(4F^2 - 3F) - 3F^2$
 $= 13F^2 - 12F$

Then $F^5 = 13F^3 - 12F^2$
 $= 13(4F^2 - 3F) - 12F^2$
 $= 40F^2 - 39F$

Similar matrix

- A & B be 2 square matrix of same order then we say $A \sim B$ i.e. A is similar to B if there exist non singular matrix P such that

$$PA = BP \text{ or } A = P^{-1}BP \quad // \text{ } P \text{ called similarity matrix}$$

- Relation of similarity is equivalence relation (reflexive, symm and transitive)

If $A \sim B$ then

- same determinant
- same char polynomial
- same eigen values
- same rank and nullity
- same trace

Diagonalization

Diagonalization - A square matrix is said to be diagonalizable if it is similar to diagonal matrix

Conditions: $A_{n \times n}$ is diagonalizable if it has n LI eigen vectors

All eigen values distinct

Q Diagonalize $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

Ans **Step 1:** Find eigen values $\Rightarrow \lambda = 0, 2, 3$

Step 2: See if matrix is diagonalizable

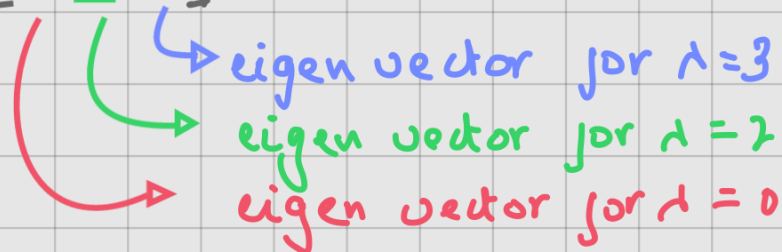
$$AM(0) = GM(0) = 1$$

$$AM(2) = GM(2) = 1$$

$$AM(3) = GM(3) = 1$$

Step 3: Find P ie matrix of eigen values

$$P = \begin{bmatrix} \text{red box} & \text{green box} & \text{blue box} \end{bmatrix}$$


eigen vector for $\lambda = 3$
eigen vector for $\lambda = 2$
eigen vector for $\lambda = 0$

Step 4: Find $P^{-1}AP$ ie diagonal matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

It will turn out to be a diagonal matrix with eigen values as diagonal elements

Repeated eigen values

Q $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

Ans $\lambda = 8, -1, -1$

for $\lambda = 8$ eigen vector = $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

for $\lambda = -1$ we get $2x_1 + x_2 + 2x_3 = 0$

If $x_1 = 1, x_3 = 0 \Rightarrow x_2 = -2$, eigen vector =

If $x_1 = 0, x_3 = 1 \Rightarrow x_2 = -2$, eigen vector =

$$\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

they are LI
hence matrix
diagonalizable

find P & diagonal matrix

Orthogonal matrix

- A square matrix is orthogonal matrix if

$$AA^T = I \quad \text{or} \quad A^{-1} = A^T$$

- Determinant of orthogonal matrix is ± 1 or -1

Proof: $|AA^T| = |I|$

$$|A||A^T| = 1$$

$$|A|^2 = 1$$

$$|A| = \pm 1$$

- Transpose of orthogonal matrix is orthogonal

Proof:

given $AA^T = I$ // definition

consider $(A^T)(A^T)^T$

$$= A^T A$$

$$= I$$

hence A^T is orthogonal

- Inverse of orthogonal matrix is orthogonal

Proof:

Consider $(A^{-1})(A^{-1})^T$
 $(A^T)(A^T)^T$
 $(A^T A)^{-1}$
 $(I)^{-1}$
 I

$$\parallel (AB)^T = B^T A^T$$

• If A & B are orthogonal then $A+B$ is not orthogonal
 Proof

$$\begin{aligned} \text{Consider } (A+B)(A+B)^T &= (A+B)(A^T+B^T) \\ &= AA^T + AB^T + BA^T + BB^T \\ &= I + AB^T + BA^T + I \\ &= 2I + AB^T + BA^T \end{aligned}$$

$\neq I$ hence not $A+B$ is not orthogonal
 will be orthogonal if $AB^T + BA^T = -I$

• Product of 2 orthogonal matrix is orthogonal
 Proof:

$$\begin{aligned} AB(AB)^T &= AB B^T A^T \\ &= A I A^T \\ &= A A^T \\ &= I \end{aligned}$$

hence AB is orthogonal

• If A is orthogonal A^n is also orthogonal
 Proof:

$$\begin{aligned} \text{consider } \hat{A}(\hat{A})^T &= \hat{A}(\hat{A}^T)^n \\ &= (\hat{A}^T \hat{A})^n \\ &= I^n \\ &= I \end{aligned}$$

$$\hat{A}(\hat{A})^T = \hat{A} \hat{A}^T \text{ if } AB = BA$$

prove by induction

hence $\hat{A}$ is orthogonal

• Dot product of any 2 row vector or any 2 column vector is 0

• All identity matrix are orthogonal matrix

Companion matrix

If char polynomial = $\hat{x} + a_{n-1}x^{n-1} - \dots - a_0$ then matrix corresponding to this char polynomial is called companion matrix A

$$A = \begin{bmatrix} \overbrace{0 \ 0 \ \dots \ 0}^{\text{All 0}} & \underbrace{-a_0}_{\text{with -ve sign}} \\ \underbrace{1 \ 0 \ \dots \ 0}_{\text{Identity}} & \underbrace{-a_1}_{\text{with -ve sign}} \\ 0 \ 1 \ \dots \ 0 & -a_2 \\ \vdots & \vdots \\ 0 \ 0 \ \dots \ 1 & -a_{n-1} \end{bmatrix}$$

eg if char polynomial is $= x^3 + 3x^2 + 2x + 4$
then

$$A = \begin{bmatrix} \overbrace{0 \ 0}^{\text{All 0}} & \underbrace{-4}_{\text{with -ve sign}} \\ \underbrace{1 \ 0}_{\text{Identity}} & \underbrace{-2}_{\text{with -ve sign}} \\ 0 \ 1 & -3 \end{bmatrix}$$

Properties of matrix

- $(AB)^{-1} \neq A^{-1}B^{-1}$ as $(AB)^{-1} = (AB)(AB)(AB) \dots$
- $(AB)^{-1} = B^{-1}A^{-1}$ if $AB = BA$

Properties of diagonal matrix

$$A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{bmatrix}$$

$$\cdot AB = BA = \begin{bmatrix} xp & 0 & 0 \\ 0 & yq & 0 \\ 0 & 0 & zr \end{bmatrix}$$

$$\cdot A^{\wedge} = \begin{bmatrix} x^{\wedge} & 0 & 0 \\ 0 & y^{\wedge} & 0 \\ 0 & 0 & z^{\wedge} \end{bmatrix}$$

$$\cdot A^{-1} = \begin{bmatrix} 1/x & 0 & 0 \\ 0 & 1/y & 0 \\ 0 & 0 & 1/z \end{bmatrix}$$

$$\cdot \text{Eigen values of } A = x, y, z$$

$$\cdot \text{Determinant of } A = xyz$$

$$\cdot A = A^T$$