

Classical Mechanics J. B. Tatum

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CHAPTER 1 CENTRES OF MASS 1.1 Introduction, and some definitions.
 This chapter deals with the calculation of the positions of the centres of mass
 of various bodies. We start with a brief explanation of the meaning of centre
 of mass, centre of gravity and centroid, and a very few brief sentences on
 their physical significance. Many students will have seen the use of calculus
 in calculating the positions of centres of mass, and we do this for Plane areas
 i for which the equation is given in x-y coordinates; ii for which the equation
 is given in polar coordinates. Plane curves i for which the equation is given in
 x-y coordinates; ii for which the equation is given in polar coordinates. Three
 dimensional figures such as solid and hollow hemispheres and cones. There
 are some figures for which interesting geometric derivations can be done
 without calculus; for example, triangular laminas, and solid tetrahedra,
 pyramids and cones. And the theorems of Pappus allow you to find the
 centres of mass of semicircular laminas and arcs in your head with no
 calculus. First, some definitions. Consider several point masses in the x-y
 plane: m at (x, y) m_2 at (x_2, y_2) etc. The centre of mass is a point $(\bar{x}, \bar{y})$ whose
 coordinates are defined by $\sum m_i x_i = \bar{x} \sum m_i$ $\sum m_i y_i = \bar{y} \sum m_i$ where M is the total mass
 $\sum m_i$. The sum $\sum m_i x_i$ is the first moment of mass with respect to the y axis.
 The sum $\sum m_i y_i$ is the first moment of mass with respect to the x axis.

If the masses are distributed in three dimensional space, with m at (x, y, z) , etc., the centre of mass is a point $(\bar{x}, \bar{y}, \bar{z})$ such that $\sum m_i x_i = M \bar{x}$, $\sum m_i y_i = M \bar{y}$, $\sum m_i z_i = M \bar{z}$. In this case, $\sum m_i x_i$, $\sum m_i y_i$, $\sum m_i z_i$ are the first moments of mass with respect to the y - z , z - x and x - y planes respectively. In either case we can use vector notation and suppose that $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are the position vectors of m_1, m_2, m_3 with respect to the origin, and the centre of mass is a point whose position vector $\mathbf{P}$ is defined by $\mathbf{P} = \frac{\sum m_i \mathbf{r}_i}{M}$. In this case the sum is a vector sum and $\sum m_i \mathbf{r}_i$, a vector quantity, is the first moment of mass with respect to the origin. Its scalar components in the two dimensional case are the moments with respect to the axes; in the three dimensional case they are the moments with respect to the planes. Many early books, and some contemporary ones, use the term "centre of gravity". Strictly the centre of gravity is a point whose position is defined by the ratio of the first moment of weight to the total weight. This will be identical to the centre of mass provided that the strength of the gravitational field g (or gravitational acceleration) is the same throughout the space in which the masses are situated. This is usually the case, though it need not necessarily be so in some contexts. For a plane geometrical figure, the centroid or centre of area, is a point whose position is defined as the ratio of the first moment of area to the total area. This will be the same as the position of the centre of mass of a plane lamina of the same size and shape provided that the lamina is of uniform surface density. Calculating the position of the centre of mass of various figures could be considered as merely a make-work mathematical exercise. However, the centres of gravity, mass and area have important applications in the study of mechanics. For example, most students at one time or another have done problems in static equilibrium, such as a ladder leaning against a wall. They will have dutifully drawn vectors indicating the forces on the ladder at the ground and at the wall, and a vector indicating the weight of the ladder. They will have drawn this as a single arrow at the centre of gravity of the ladder as if the entire weight of the ladder could be "considered to act" at the centre of gravity. In what sense can we take this liberty and "consider all the weight as if it were concentrated at the centre of gravity"? In fact

the ladder consists of many point masses (atoms) all along its length. One of the equilibrium conditions is that there is no net torque on the ladder. The definition of the centre of gravity is such that the sum of the moments of the

weights of all the atoms about the base of the ladder is equal to the total weight times the horizontal distance to the centre of gravity, and it is in that sense that all the weight "can be considered to act" there. Incidentally, in this example, "centre of gravity" is the correct term to use. The distinction would be important if the ladder were in a nonuniform gravitational field. In dynamics, the total linear momentum of a system of particles is equal to the total mass times the velocity of the centre of mass. This may be "obvious", but it requires formal proof, albeit one that follows very quickly from the definition of the centre of mass. 1.2 Likewise the kinetic energy of a rigid body in two dimensions equals $\frac{1}{2}MV^2 + \frac{1}{2}I\omega^2$, where M is the total mass, V the speed of the centre of mass, I the rotational inertia and ω the angular speed, both around the centre of mass. Again it requires formal proof, but in any case it furnishes us with another example to show that the calculation of the positions of centres of mass is more than merely a make-work mathematical exercise and that it has some physical significance. If a vertical surface is immersed under water (e.g. a dam wall) it can be shown that the total hydrostatic force on the vertical surface is equal to the area times the pressure at the centroid. This requires proof (readily deduced from the definition of the centroid and elementary hydrostatic principles), but it is another example of a physical application of knowing the position of the centroid.

1.2 Plane triangular lamina Definition: A median of a triangle is a line from a vertex to the mid point of the opposite side. Theorem I. The three medians of a triangle are concurrent (meet at a single, unique point) at a point that is two-thirds of the distance from a vertex to the mid point of the opposite side. Theorem II. The centre of mass of a uniform triangular lamina (or the centroid of a triangle) is at the meet of the medians. The proof of I can be done with a nice vector argument (figure I. 1): Let $\mathbf{A}$, $\mathbf{B}$ be the vectors $\mathbf{OA}$, $\mathbf{OB}$. Then $\mathbf{A} + \mathbf{B}$ is the diagonal of the parallelogram of which $\mathbf{OA}$ and $\mathbf{OB}$ are two sides, and the position vector of the point C is $\frac{1}{3}(\mathbf{A} + \mathbf{B})$. To get C_2 , we see that: $\mathbf{M} = \frac{1}{3}(\mathbf{A} + \mathbf{B})$ $\mathbf{C}_2 = \frac{1}{3}(\mathbf{A} + \mathbf{B})$ $\mathbf{C} = \frac{1}{3}(\mathbf{A} + \mathbf{B})$ $\mathbf{C} = \frac{1}{3}(\mathbf{A} + \mathbf{B})$

FIGURE I. 1 Thus the points C and C_2 are identical, and the same would be true for the third median, so Theorem I is proved. Now consider an elemental slice as in figure 1.2. The centre of mass of the slice is at its mid-point. The same is true of any similar slices parallel to it. Therefore the centre of mass is on the locus of the mid-points - i.e. on a

median. Similarly it is on each of the other medians, and Theorem II is proved. FIGURE 1.2 That needed only some vector geometry. We now move on to some calculus.

1.3 Plane areas. Plane areas in which the equation is given in x-y coordinates
 FIGURE 1.3 We have a curve $y = y(x)$ (figure 1.3) and we wish to find the position of the centroid of the area under the curve between $x = a$ and $x = b$. We consider an elemental slice of width dx at a distance x from the y axis. Its area is ySx , and so the total area is $A = \int y dx$. 1.3.1 The first moment of area of the slice with respect to the y axis is $xySx$, and so the first moment of the entire area is $I_y = \int xy dx$. Therefore $\bar{x} = \frac{I_y}{A}$. 1.3.2 $I_y = \int xy dx$

6 For y we notice that the distance of the centroid of the slice from the x axis is $y/2$. therefore the first moment of the area about the x axis is $\frac{1}{2} y^2 dx$, and Therefore $I_x = \int \frac{1}{2} y^2 dx$. 1.3.3 2:/ Example. Consider a semicircular lamina, $x^2 + y^2 = a^2$, $x > 0$, see figure 1.4: FIGURE 1.4 We are dealing with the parts both above and below the x axis, so the area of the semicircle is $A = \int_{-a}^a y dx$ and the first moment of area is $I_y = \int xy dx$. You should find $\bar{x} = \frac{4a}{3\pi} = 0.4244a$. Now consider the lamina $x^2 + y^2 = a^2$, $y > 0$ (figure 1.5): FIGURE 1.5

The area of the elemental slice this time is $y dx$ (not $2y dx$), and the integration limits are from $-a$ to $+a$. To find $\bar{x}$, use equation 1.3.3, and you should get $\bar{x} = 0.4244a$. Plane areas in which the equation is given in polar coordinates. / / / / FIGURE 1.6 We consider an elemental triangular sector (figure 1.6) between θ and $\theta + d\theta$. the triangle is r and the "base" is $r d\theta$. The area of the triangle is $\frac{1}{2} r^2 d\theta$. Therefore the whole area $A = \int \frac{1}{2} r^2 d\theta$. The "height" of 1.3.4 The horizontal distance of the centroid of the elemental sector from the origin (more correctly, from the "pole" of the polar coordinate system) is $r \cos \theta$. The first moment of area of the sector with respect to the y axis is $r \cos \theta \times \frac{1}{2} r^2 d\theta$: $\frac{1}{2} r^3 \cos \theta d\theta$ so the first moment of area of the entire figure between $\theta = 0$ and $\theta = 2\pi$ is

Therefore $\bar{x} = \frac{1}{A} \int \frac{1}{2} r^3 \cos \theta d\theta$. 3 $J = \int r^3 \cos \theta d\theta$ 3 r : $d\theta$ 1.3.5 Similarly $y = 2f, r$ 3 $\sin \theta d\theta$ 3 $I = \int r^3 \sin \theta d\theta$ 1.3.6 Example: Consider the semicircle $r = a$, $\theta = -\pi/2$ to $+\pi/2$. $f = \frac{1}{2} \int_{-\pi/2}^{+\pi/2} 2a \cos \theta d\theta = 2a \sin \theta \Big|_{-\pi/2}^{+\pi/2} = 4a$ 1.3.7 3 $[\pi/2]$ $d\theta$ 3 $c/2$ The reader should now try to find the position of the centroid of a circular sector (slice of pizza!) of angle $2c$. The integration limits will be $-c$ to $+c$. When you

arrive at a formula (which you should keep in a notebook for future reference), check that it goes to $4a/(3c)$ if $c = c/2$, and to $2a/3$ if $c = 0$. 1.4 Plane curves Plane curves in which the equation is given in x-y coordinates

FIGURE 1.7 a b x

Figure 1.7 shows how an elemental length ds is related to the corresponding increments in x and y : $ds = \sqrt{dx^2 + dy^2}$.

1.4.1 Consider a wire of mass per unit length (linear density) L bent into the shape $y = y(x)$ between $x = a$ and $x = b$. The mass of an element ds is $L ds$, so the total mass is $M = \int_a^b L ds = L \int_a^b \sqrt{1 + (dy/dx)^2} dx$. 1.4.2 The first moments of mass about the y - and x -axes are respectively $\int_a^b x \sqrt{1 + (dy/dx)^2} dx$ and $\int_a^b y \sqrt{1 + (dy/dx)^2} dx$. 1.4.3 If the wire is uniform and is therefore not a function of x or y , can come outside the integral signs in equations 1.12 and 1.13, and we hence obtain $M = L \int_a^b \sqrt{1 + (dy/dx)^2} dx$, 1.4.4 the denominator in each of these expressions merely being the total length of the wire. Example: Consider a uniform wire bent into the shape of the semicircle $x^2 + y^2 = a^2$, $x > 0$. First, it might be noted that one would expect $> 0.4244a$ (the value for a plane semicircular lamina). The length (i.e. the denominator in equation 1.4.4) is just πa . Since there are, between x and $x + \Delta x$, two elemental lengths to account for, one above and one below the x axis, the numerator of the first of equation 1.4.4 must be

$2 \int_0^a x \sqrt{1 + (dy/dx)^2} dx$ (a - x: The first moment of length of the entire semicircle is From this point the student is left to his or her own devices to derive $Z = 2a/c = 0.6333a$. Plane curves in which the equation is given in polar coordinates. FIGURE 1.8 Figure 1.8 shows how an elemental length ds is related to the corresponding increments in r and θ :

11 1.4.5 The mass of the curve (between $\theta = \theta_1$ and $\theta = \theta_2$) is $M = \int_{\theta_1}^{\theta_2} L r d\theta$. The first moments about the y - and x -axes are (recalling that $x = r \cos \theta$ and $y = r \sin \theta$) $\int_{\theta_1}^{\theta_2} L r^2 \cos \theta d\theta$ and $\int_{\theta_1}^{\theta_2} L r^2 \sin \theta d\theta$. If X is not a function of r or θ , we obtain $M = L \int_{\theta_1}^{\theta_2} r^2 d\theta$. 1.4.6 where L is the length of the wire. Example: Again consider the uniform wire of figure 1.8 bent into the shape of a semicircle. The equation in polar coordinates is simply $r = a$, and the integration limits are $\theta = -\pi/2$ to $\theta = \pi/2$. The length is πa . Thus: $\int_{-\pi/2}^{\pi/2} [0, \pi/2] 2a \cos \theta d\theta = 2a \int_{-\pi/2}^{\pi/2} \cos \theta d\theta = 4a$. The reader should now find the position of the centre of mass of a wire bent into the arc of a circle of angle

2or. The expression obtained should go to $2\pi pc$ as ϕ goes to $c/2$, and to a as ϕ goes to zero.

12 1.5 Summary of the formulas for plane laminas and curves SUMMARY
 Uniform Plane Lamina $y = y(x)$ $r = r(\theta)$, $b = 2[r \cos \theta]$, $a = 2[r \sin \theta]$ $dO = b dx$
 $Y = \frac{1}{2} \int_{-b}^b y^2 dx$ Uniform Plane Curve $y = y(x)$ $r = r(\theta)$ $b = 2 \int_{-b/2}^{b/2} y dx$
 $+ dO$

13 1.6 The Theorems of Pappus. (Pappus Alexandrinus, Greek mathematician, approximately 3rd or 4th century AD.) If a plane area is rotated about an axis in its plane, but which does not cross the area, the volume swept out equals the area times the distance moved by the centroid. II. If a plane curve is rotated about an axis in its plane, but which does not cross the curve, the area swept out equals the length times the distance moved by the centroid. These theorems enable us to work out the volume of a solid of revolution if we know the position of the centroid of a plane area, or vice versa; or to work out the area of a surface of revolution if we know the position of the centroid of a plane curve or vice versa. It is not necessary that the plane or the curve be rotated through a full 360° . We prove the theorems first. We then follow with some examples. FIGURE I. 9 A y

14 Consider an area A in the zx plane (figure 1.9), and an element δA within the area at a distance x from the z axis. Rotate the area through an angle θ about the z axis. The length of the arc traced by the element δA in moving through an angle θ is $x\theta$, so the volume swept out by δA is $x\theta \delta A$. The volume swept out by the entire area is $\theta \int x dA$. But the definition of the centroid of A is such that its distance from the z axis is given by $\bar{x} = \frac{1}{A} \int x dA$. Therefore the volume swept out by the area is $\theta \bar{x} A$. But $\theta \bar{x}$ is the distance moved by the centroid, so the first theorem of Pappus is proved. FIGURE I. 10 y x Consider a curve of length L in the zx plane (figure I. 10), and an element δs of the curve at a distance x from the z axis. Rotate the curve through an angle θ about the z axis. The length of the arc traced by the element δs in moving through an angle θ is $x\theta$, so the area swept out by δs is $x\theta \delta s$. The area swept out by the entire curve is $\theta \int x ds$. But the definition of the centroid is such that its distance from the z axis is given by $\bar{x} = \frac{1}{L} \int x ds$. Therefore the area swept out by the curve is $\theta \bar{x} L$. But $\theta \bar{x}$ is the distance moved by the centroid, so the second theorem of Pappus is proved.

15 Applications of the Theorems of Pappus. Rotate a plane semicircular figure of area a through 360° about its diameter. The volume swept out is $4\pi a \bar{y}$, and the distance moved by the centroid is $2\pi \bar{y}$. Therefore by the theorem of Pappus, $V = 4\pi a \bar{y}$. Rotate a plane semicircular arc of length s through 360° about its diameter. Use a similar argument to show that $V = 2\pi \bar{y} s$. FIGURE I. 11 Consider a right-angled triangle, height h , base a (figure I. 11). Its centroid is at a distance $a/3$ from the height h . The area of the triangle is $ah/2$. Rotate the triangle through 360° about h . The distance moved by the centroid is $2\pi a/3$. The volume of the cone swept out is $ah/2$ times $2\pi a/3$, equals $a^2h/3$. Now consider a line of length l inclined at an angle θ to the y axis (figure I. 12). Its centroid is at a distance $l/3 \sin \theta$ from the y axis. Rotate the line through 360° about the y axis. The distance moved by the centroid is $2\pi l/3 \sin \theta$. The surface area of the cone swept out is $l \times 2\pi l \sin \theta = \pi l^2 \sin \theta$.

16 FIGURE I. 12 The centre of a circle of radius b is at a distance a from the y axis. It is rotated through 360° about the y axis to form a torus (figure I. 13). Use the theorems of Pappus to show that the volume and surface area of the torus are, respectively, $2\pi ab^2$ and $4\pi ab$. FIGURE I. 13

17 1.7 Uniform solid tetrahedron, pyramid and cone. Definition. A median of a tetrahedron is a line from a vertex to the centroid of the opposite face. Theorem I. The four medians of a tetrahedron are concurrent at a point $3/4$ of the way from a vertex to the centroid of the opposite face. Theorem II. The centre of mass of a uniform solid tetrahedron is at the meet of the medians. Theorem I can be derived by a similar vector geometric argument used for the plane triangle. It is slightly more challenging than for the plane triangle, and it is left as an exercise for the reader. I draw two diagrams (figure 1.14). One shows the point C_1 that is $3/4$ of the way from the vertex A to the centroid of the opposite face. The other shows the point C_2 that is $3/4$ of the way from the vertex B to the centroid of its opposite face. You should be able to show that $C = (A+B+D)/4$. FIGURE I. 14

18 In fact this suffices to prove Theorem I, because, from the symmetry between A , B and D , one is bound to arrive at the same expression for the three-quarter way mark on any of the four medians. But for reassurance you should try to show, from the second figure, that $C_2 = (A+B+D)/4$. The

argument for Theorem II is easy, and is similar to the corresponding argument for plane triangles. Pyramid. A right pyramid whose base is a regular polygon (for example, a square) can be considered to be made up of several tetrahedra stuck together. Therefore the centre of mass is $3/4$ of the way from the vertex to the mid point of the base. Cone. A right circular cone is just a special case of a regular pyramid in which the base is a polygon with an infinite number of infinitesimal sides. Therefore the centre of mass of a uniform right circular cone is $3/4$ of the way from the vertex to the centre of the base. We can also find the position of the centre of mass of a solid right circular cone by calculus. We can find its volume by calculus, too, but we'll suppose that we already know, from the theorem of Pappus, that the volume is $\frac{1}{3} \times \text{base} \times \text{height}$. of Pappus, that the volume is $\frac{1}{3} \times \text{base} \times \text{height}$.  FIGURE I. 15

19 Consider the cone in figure I. 15, generated by rotating the line $y = ax/h$ (between $x = 0$ and $x = h$) through 360° about the x axis. The radius of the elemental slice of thickness Δx at x is ax/h . Its volume is $\pi (ax/h)^2 \Delta x$. Since the volume of the entire cone is $\pi a^2 h^3 / 3$, the mass of the slice is $\frac{M}{V} \times \pi a^2 x^2 \Delta x$. The first moment of mass of the elemental slice with respect to the y axis is $\frac{M}{V} \times \pi a^2 x^3 \Delta x / 3$. The position of the centre of mass is therefore $\frac{1}{3} h$. Hollow cone. The surface of a hollow cone can be considered to be made up of an infinite number of infinitesimally slender isosceles triangles, and therefore the centre of mass of a hollow cone (without base) is $2/3$ of the way from the vertex to the midpoint of the base. 1.9 Hemispheres. Uniform solid hemisphere Figure 1.4 will serve. The argument is exactly the same as for the cone. The volume of the elemental slice is $\pi y^2 \Delta x = \pi (a^2 - x^2) \Delta x$, and the volume of the hemisphere is $2\pi a^3 / 3$, so the mass of the slice is $\frac{M}{V} \times \pi (a^2 - x^2) \Delta x$ where M is the mass of the hemisphere. The first moment of mass of the elemental slice is x times this, so the position of the centre of mass is $3a/8$.

20 Hollow hemispherical shell. We may note to begin with that we would expect the centre of mass to be further from the base than for a uniform solid hemisphere. Again, figure 1.4 will serve. The area of the elemental annulus is $2\pi a x$ (NOT $2\pi x^2$!) and the area of the hemisphere is $2\pi a^2$. Therefore the mass of the elemental annulus is $\frac{M}{A} \times 2\pi a x = \frac{M}{a} x$. The first moment of mass of the annulus is x times this, so the position of the centre of mass is $\frac{8a}{15}$. 1.10 Summary. SUMMARY Triangular lamina: $2/3$ of way from

vertex to midpoint of opposite side Solid Tetrahedron, Pyramid, Cone: $\frac{3}{4}$ of way from vertex to midpoint of opposite face. Hollow cone: $\frac{2}{3}$ of way from vertex to midpoint of base. Semicircular lamina: $\frac{4a}{3}$ Lamina in form of a sector of a circle, angle 2α : $\frac{2a \sin \alpha}{3\alpha}$ Semicircular wire: $\frac{2a}{\pi}$ Wire in form of an arc of a circle, angle 2α : $\frac{a \sin \alpha}{\alpha}$ Solid hemisphere: $\frac{3a}{8}$ Hollow hemisphere: $\frac{a}{2}$

1 CHAPTER 2 MOMENT OF INERTIA 2.1 Definition of Moment of Inertia

Consider a straight line (the "axis") and a set of point masses $m_1, m_2, m_3, \dots$ such that the distance of the mass m_i from the axis is r_i . The quantity $m_i r_i^2$ is the second moment of the i th mass with respect to (or "about") the axis, and the sum $\sum m_i r_i^2$ is the second moment of mass of all the masses with respect to the axis. Apart from some subtleties encountered in general relativity, the word "inertia" is synonymous with mass - the inertia of a body is merely the ratio of an applied force to the resulting acceleration. Thus $\sum m_i r_i^2$ can also be called the second moment of inertia. The second moment of inertia is discussed so much in mechanics that it is usually referred to as just "the" moment of inertia. In this chapter we shall consider how to calculate the (second) moment of inertia for different sizes and shapes of body, as well as certain associated theorems. But the question should be asked: "What is the purpose of calculating the squares of the distances of lots of particles from an axis, multiplying these squares by the mass of each, and adding them all together? Is this merely a pointless make-work exercise in arithmetic? Might one just as well, for all the good it does, calculate the sum $\sum m_i r_i$." Does $\sum m_i r_i^2$ have any physical significance. 2.2 Meaning of Rotational Inertia. If a force acts on a body, the body will accelerate. The ratio of the applied force to the resulting acceleration is the inertia (or mass) of the body. If a torque acts on a body that can rotate freely about some axis, the body will undergo an angular acceleration. The ratio of the applied torque to the resulting angular acceleration is the rotational inertia of the body. It depends not only on the mass of the body, but also on how that mass is distributed with respect to the axis. Consider the system shown in figure II. 1. 0 r O FIGURE II. 1 TF

A particle of mass m is attached by a light (i.e. zero or negligible mass) arm of length r to a point at O , about which it can freely rotate. A force F is applied, and the mass consequently undergoes a linear acceleration F/m . The angular acceleration is then $F/(mr)$. Also, the torque is Fr . The ratio of the

applied torque to the angular acceleration is therefore $\tau = I\alpha$. Thus the rotational inertia is the second moment of inertia. Rotational inertia and (second) moment of inertia are one and the same thing, except that rotational inertia is a physical concept and moment of inertia is its mathematical representation.

2.3 Moments of inertia of some simple shapes.

A student may well ask: "For how many different shapes of body must I commit to memory the formulas for their moments of inertia?" I would be tempted to say: "None". However, if any are to be committed to memory, I would suggest that the list to be memorized should be limited to those few bodies that are likely to be encountered very often (particularly if they can be used to determine quickly the moments of inertia of other bodies) and for which it is easier to remember the formulas than to derive them. With that in mind I would recommend learning no more than five. In the following, each body is supposed to be of mass m and rotational inertia I .

1. A rod of length l about an axis through the middle, and at right angles to the rod: $I = \frac{1}{12}ml^2$ 2.3.1
2. A uniform circular disc of radius a about an axis through the centre and perpendicular to the plane of the disc: $I = \frac{1}{2}ma^2$ 2.3.2
3. A uniform right-angled triangular lamina about one of its shorter sides - i.e. not the hypotenuse. The other not-hypotenuse side is of length a : $I = \frac{1}{6}ma^2$ 2.3.3
4. A uniform solid sphere of radius a about an axis through the centre. $I = \frac{2}{5}ma^2$ 2.3.4
5. A uniform spherical shell of radius a about an axis through the centre. $I = \frac{2}{3}ma^2$ 2.3.5

I shall now derive the first three of these by calculus. The derivations for the spheres will be left until later.

1. Rod, length l (Figure 11.2)  The mass of an element dx at a distance x from the middle of the rod is $m \frac{dx}{l}$ and its second moment of inertia is $\frac{1}{2}m \frac{dx}{l} x^2$. The moment of inertia of the entire rod is $\int_{-l/2}^{l/2} \frac{1}{2}m \frac{dx}{l} x^2 = \frac{1}{12}ml^2$.

2. Disc, radius a . (Figure 11.3)  The area of an elemental annulus, radii r , $r + dr$ is $2\pi r dr$.

The area of the entire disc is πa^2 . Therefore the mass of the annulus is and its second moment of inertia is $2\pi r^3 dr$ and $\frac{1}{2}2\pi r^3 dr$ respectively. Integrating from $r=0$ to $r=a$ gives $I = \frac{1}{2}\pi a^4$ and $m = \pi a^2 \rho$, so $I = \frac{1}{2}ma^2$.

3. Right-angled triangular lamina. (Figure 11.4)  The equation to the hypotenuse is $y = b(1 - x/a)$. The area of the elemental strip is $y dx = b(1 - x/a) dx$ and the area of the entire triangle is $ab/2$. Therefore the mass of the elemental strip is $m \frac{y dx}{ab/2} = \frac{2m}{a} (1 - x/a) dx$ and its second moment of inertia is $\frac{1}{3} \frac{2m}{a} (1 - x/a) x^3$. Integrating from $x=0$ to $x=a$ gives $I = \frac{1}{6}ma^2$.

is $2mx^2(a-x)$ The second moment of inertia of the entire triangle is the integral of this from $x = 0$ to $x = a$, which is $ma^2/6$.

Uniform circular lamina about a diameter. For the sake of one more bit of integration practice, we shall now use the same argument to show that the moment of inertia of a uniform circular disc about a diameter is $ma^2/4$.

However, we shall see later that it is not necessary to resort to integral calculus to arrive at this result, nor is it necessary to commit the result to memory. In a little while it will become immediately apparent and patently obvious, with no calculation, that the moment of inertia must be $ma^2/4$.

However, for the time being, let us have some more calculus practice. See figure II.5. FIGURE 11.5 The disc is of radius a , and the area of the elemental strip is $2ybx$. But y and x are related through the equation to the circle, which is $y = (a^2 - x^2)^{1/2}$. Therefore the area of the strip is $2(a^2 - x^2)^{1/2}x$. The second moment of inertia about the y -axis is $2\int_0^a x^2(a^2 - x^2)^{1/2} dx$, where σ is the surface density $m/\pi a^2$. For the entire disc, we integrate from $x = -a$ to $x = +a$, or, if you prefer, from $x = 0$ to $x = a$ and then double it. The result $ma^2/4$ should follow. If you need a hint about how to do the integration, let $x = a \cos \theta$ (which it is, anyway), and be sure to get the limits of integration with respect to θ right. The moment of inertia of a uniform semicircular lamina of mass m and radius a about its base, or diameter, is also $ma^2/4$, since the mass distribution with respect to rotation about the diameter is the same.

2.4 Radius of gyration. The second moment of inertia of any body can be written in the form mk^2 . Thus, for the rod, the disc (about an axis perpendicular to its plane), the triangle and the disc (about a diameter), k has the values $[a/2, a/\sqrt{2}, a/2, a/\sqrt{2}]$ respectively. k is called the radius of gyration. If you were to concentrate all the mass of a body at its radius of gyration, its moment of inertia would remain the same.

2.5 Plane Laminas, and Mass Points Distributed in a Plane. I start by considering two very important theorems, namely, the Parallel Axes Theorem and the Perpendicular Axes Theorem. The former can also be used with solid bodies and mass distributions in three-dimensional space, but it is important to understand that the latter is applicable only to plane laminas and masses distributed in a plane. Parallel axes theorem $I = I_c + md^2$ Perpendicular axes theorem $I_z = I_x + I_y$ FIGURE 11.6

7 In figure 11.6, $m_1, m_2, m_3, \dots$ are several point masses distributed in a plane. I have drawn two sets of coordinate axes. The origin of one of them is at the centre of mass C. The other axes are parallel to the first, but they have their origin at a point P. The coordinates of m_i with respect to the axes through C are (x_i, y_i) . The coordinates of P with respect to the axes through C are $(\bar{x}, \bar{y})$. The coordinates of m_i with respect to the axes through P are $(x_i - \bar{x}, y_i - \bar{y})$. Let $B_C = \sum m_i x_i^2$ and $B_P = \sum m_i (x_i - \bar{x})^2$. 2.5.1 and 2.5.2 Then $B_P = \sum m_i x_i^2 - 2\bar{x} \sum m_i x_i + \bar{x}^2 \sum m_i$. 2.5.3 The first term on the right hand side is B_C . The expression $\sum m_i x_i$ is the first moment of mass about the centre of mass and is zero. The sum $\sum m_i$ is the total mass M. Therefore $B_P = B_C + M\bar{x}^2$. 2.5.4 Similarly with respect to the moments about the horizontal axes: $A = A_C + M\bar{y}^2$. This is the Parallel Axes Theorem. In words, the moment of inertia about an arbitrary axis is equal to the moment of inertia about a parallel axis through the centre of mass plus the total mass times the square of the distance between the parallel axes. As mentioned above, the theorem holds also for masses distributed in three-dimensional space. Perpendicular Axes Theorem. Y FIGURE II.7 m_1, m_2, m_3

8 Figure 11.7 shows some point masses distributed in the xy plane, the z axis being perpendicular to the plane of the paper. The moments of inertia about the x, y and z axes are denoted respectively by A, B and C. The distance of m_i from the z axis is $(x_i^2 + y_i^2)^{1/2}$. Therefore the moment of inertia of the masses about the z axis is $C = A + B$. 2.5.5 That is to say: 2.5.6 This is the Perpendicular Axes Theorem. Note very carefully that, unlike the parallel axes theorem, this theorem applies only to plane laminae and to point masses distributed in a plane. Examples of the Use of the Parallel and Perpendicular Axes Theorems. From section 2.3 we know the moments of inertia of discs, rods and triangular laminae. We can make use of the parallel and perpendicular axes theorems to write down the moments of inertia of most of the following examples almost by sight, with no calculus. Hoop and discs, radius a. $I_C = \frac{1}{2}ma^2$ $I_A = I_B = \frac{1}{4}ma^2$ Rods, length 2l. $I_C = \frac{1}{12}ml^2$ $I_A = I_B = \frac{1}{3}ml^2$

Rectangular laminae, sides 2a and 2b; $a > b$. $I_C = \frac{1}{12}m(a^2 + b^2)$ Square laminae, side 2a. $I_C = \frac{1}{12}ma^2$ $I_A = I_B = \frac{1}{4}ma^2$ Triangular laminae. $I_C = \frac{1}{12}ma^2$ $I_A = I_B = \frac{1}{6}m(a^2 + b^2)$ $I_C = \frac{1}{12}m(a^2 + b^2)$

10 b, c, a $I = \frac{1}{12}m(a^2 + b^2 + c^2)$ $I = \frac{1}{12}m(a^2 + b^2 + c^2)$ $I = \frac{1}{12}m(a^2 + b^2 + c^2)$ $I = \frac{1}{12}m(a^2 + b^2 + c^2)$

+c:) = mOc:-a:) = $\frac{1}{2}m(a^2 + 2.6$ Three-dimensional solid figures. Spheres, cylinders, cones Sphere, mass m, radius a. The volume of an elemental cylinder of radii x, x+dx, height 2y is $4\pi x^2 y dx$. Its mass is $\frac{4\pi x^2 y dx}{a^3} m$. Its second moment of inertia is $2\pi x^4 y dx$. The second moment of inertia of the entire sphere is

11 $\frac{1}{2} \pi a^2 \int_0^a x^2 dx = \frac{1}{2} \pi a^2 \left[\frac{x^3}{3} \right]_0^a = \frac{1}{6} \pi a^3$ The moment of inertia of a uniform solid hemisphere of mass m and radius a about a diameter of its base is also $\frac{1}{6} \pi a^3$, because the distribution of mass around the axis is the same as for a complete sphere. Problem: A hollow sphere is of mass M, external radius a and internal radius x. Its rotational inertia is $0.5 Ma^2$. Show that x is given by the solution of $1 - 5x^3 + 4x^5 = 0$ and calculate x to four significant figures. (Answer = 0.6836.) Solid cylinder, mass m, radius a, length l. The mass of an elemental disc of thickness dx is $\frac{m}{l} dx$. Its moment of inertia about its diameter is $\frac{1}{2} m dx a^2$. Its moment of inertia about the dashed axis through the centre of the cylinder is $\frac{1}{2} m(a^2 + x^2) dx$. The moment of inertia of the entire cylinder is $\frac{1}{2} m(a^2 + \frac{l^2}{3})$. In a similar manner it can be shown that the moment of inertia of a uniform solid triangular prism of mass m, length l, cross section an equilateral triangle of side 2a about an axis through its centre and perpendicular to its length is $m(\frac{1}{2} a^2 + \frac{l^2}{12})$.

12 Solid cone, mass m, height h, base radius a. The mass of the elemental disc of thickness dx is $\frac{m}{h} x dx$. Its second moment of inertia about the axis of the cone is $\frac{1}{2} \frac{m}{h} x^3 dx$. But x and r are related through $y = \frac{a}{h} x$, so the moment of inertia of the elemental disk is $\frac{3}{2} \frac{m}{h} x^3 dx$. The moment of inertia of the entire cone is

13 $\frac{3}{10} ma^2$ The following, for a solid cone of mass m, height h, base radius a, are left as an exercise: $m(3a^2 + 2h^2)$ 20 2.7 Three-dimensional hollow figures. Spheres, cylinders, cones. Hollow spherical shell, mass m, radius a.

14 The area of the elemental zone is $2\pi a^2 \sin \theta d\theta$. Its mass is $\frac{2\pi a^2 \sin \theta d\theta}{4\pi a^2} m$. Its moment of inertia about the axis is $\frac{2\pi a^4 \sin^3 \theta d\theta}{3}$. The moment of inertia of the entire spherical shell is $\frac{2\pi a^4}{3} \int_0^\pi \sin^3 \theta d\theta = \frac{8\pi a^4}{15}$. This result

can be used to calculate, by integration, the moment of inertia of a solid sphere. It is left as an exercise to show that for a uniform solid sphere, the result is $\frac{2}{5}ma^2$. Using methods similar to that given for a solid cylinder, it is left as an exercise to show that the moment of inertia of an open hollow cylinder about an axis perpendicular to its length passing through its centre of mass is $m(a^2 + \frac{1}{2}l^2)$ where a is the radius and l is the length. The moment of inertia of a baseless hollow cone of mass m , base radius a , about the axis of the cone could be found by integration. However, those who have an understanding of the way in which the moment of inertia depends on the distribution of mass should readily see, without further ado, that the moment of inertia is $\frac{3}{8}ma^2$.

2.8 Torus The rotational inertias of solid and hollow toruses (large radius a , small radius b) are given below for reference and without derivation. They can be derived by integral calculus, and their derivation is recommended as a challenge to the reader. Solid torus: $\frac{5}{8}m(4a^2 + 3b^2)$ Hollow torus: $\frac{1}{2}m(2a^2 + 5b^2)$

2.9 Linear triatomic molecule $x-y-z$ Here is an interesting problem. It should be straightforward to calculate the rotational inertia of the above molecule with respect to an axis perpendicular to the molecule and passing through the centre of mass. In practice it is quite easy to measure the rotational inertia very precisely from the spacing between the lines in a molecular band in the infrared region of the spectrum. If you know the three masses (which you do if you know the atoms that make up the molecule) can you calculate the two interatomic spacings x and y ? That would require determining two unknown quantities, x and y , from a single measurement of the rotational inertia, I . Evidently that cannot be done; a second measurement is required. Can you suggest what might be done? We shall answer that shortly. In the meantime, it is an exercise to show that the rotational inertia is given by $ax^2 + 2hxy + by^2 + c = 0$, 2.9.1 where $a = m_1(m_2 + m_3)/M$ 2.9.2 $h = m_1m_2r/m_3$ 2.9.3 $b = m_3(m_1 + m_2)/M$ 2.9.4 $M = m_1 + m_2 + m_3$ 2.9.5

2.9.6 For example, suppose the molecule is the linear molecule OCS, and the three masses are 16, 12 and 32 respectively, and, from infrared spectroscopy, it is determined that the moment of inertia is 20. (For this hypothetical illustrative example, I am not concerning myself with units). In that case, equation 2.9.1 becomes $11.7x^2 + 17.0xy + 14.9y^2 - 20 = 0$. 2.9.7 We

need another equation to solve for x and y . What can be done chemically is to prepare an isotopically-substituted molecule (isotopomer) such as $aOCS$, and measure its moment of inertia from its spectrum, making the probably very justified assumption that the interatomic distances are unaffected by the isotopic substitution. This results in a second equation: $a'x + 2h'xy + b'y + c' = 0$.

2.9.8 Let's suppose that $I = 21$, and I leave it to the reader to work out the numerical values of a' , h' and b' with the stern caution to retain all the decimal places on your calculator. That is, do not round off the numbers until the very end of the calculation. You now have two equations, 2.9.1 and 2.9.8, to solve for x and y . These are two simultaneous quadratic equations, and it may be that some guidance in solving them would be helpful. I have three suggestions. Treat equation 2.9.1 as a quadratic equation in x and solve it for x in terms of y . Then substitute this in equation 2.9.8. I expect you will very soon become bored with this method and will want to try something a little less tedious. You have two equations of the form $S(x,y) = 0$, $S'(x,y) = 0$. There are standard ways of solving these iteratively by an extension of the Newton-Raphson process. This is described, for example, in section 1.9 of Chapter 1 of my Celestial Mechanics notes, and this general method for two or more nonlinear equations should be known by anyone who expects to engage in much numerical calculation. For this particular case, the detailed procedure would be as follows. This is an iterative method, and it is first necessary to make a guess at the solutions for x and y . The guesses need not be particularly good. That done, compute the following six quantities: S : $x(ax + 2hy) + by + c$ S'_x : $a(x + hy)$ S'_y : $2(hx + by)$

S'_x : $2(a'x + h'y)$ S'_y : $2(h'x + b'y)$ Here the subscripts denote the partial derivatives. Now if $x(\text{true}) = x(\text{guess}) + \delta x$ and $y(\text{true}) = y(\text{guess}) + \delta y$ the errors δx and δy can be found from the solution of $S + S'_x \delta x + S'_y \delta y = 0$ and $S'_x \delta x + S'_y \delta y = 0$. If we calculate $F = S + S'_x \delta x + S'_y \delta y$ the solutions for the errors are $F(S'_x S'_y - S'_x S'_y)$. This will enable a better guess to be made, and the procedure can be repeated until the errors are as small as desired. Generally only a very few iterations are required. If this is not the case, a programming mistake is indicated. While method 2 can be used for any nonlinear simultaneous equations, in this particular case we have two simultaneous quadratic equations, and a little familiarity with conic sections provides a rather nice method. Thus, if $S = 0$ and $S' = 0$ are equations 2.9.1 and 2.9.8 respectively. Each of these equations represents a conic section, and they intersect at four

points. We wish to find the point of intersection that lies in the all-positive quadrant - i.e. with x and y both positive. Since the two conic sections are very similar, in order to calculate where they intersect it is necessary to calculate with great accuracy. Therefore, do not round off the numbers until the very end of the calculation. Form the equation $c'S - cS' = 0$. This is also a quadratic equation representing a conic section passing through the four points. Furthermore, it has no constant term, and it therefore represents the two straight lines that pass through the four points. The equation can be factorized into two linear terms, $oq^3 = 0$, where $ot = 0$ and $3 = 0$ are the two straight lines. Choose the one with positive slope and solve it with $S = 0$ or with $S' = 0$ (or with both, as a check against arithmetic mistakes) to find x and y . In this case, the solutions are $x = 0.2529$, $y = 1.000$.

18 2.10 Pendulums In section 2.2, we discussed the physical meaning of the rotational inertia as being the ratio of the applied torque to the resulting angular acceleration. In linear motion, we are familiar with the equation $F = ma$. The corresponding equation when dealing with torques and angular acceleration is $\tau = I\alpha$. We are also familiar with the equation of motion for a mass vibrating at the end of a spring of force constant k : $m\ddot{x} = -kx$. This is simple harmonic motion of period $2\pi/\sqrt{k}$. The mechanics of the torsion pendulum is similar. The torsion constant c of a wire is the torque required to twist it through unit angle. If a mass is suspended from a torsion wire, and the wire is twisted through an angle θ , the restoring torque will be $c\theta$, and the equation of motion is $I\ddot{\theta} = -c\theta$, which is simple harmonic motion of period $2\pi/\sqrt{c/I}$. The torsion constant of a wire of circular cross-section, by the way, is proportional to its shear modulus, the fourth power of its radius, and inversely as its length. The derivation of this takes a little trouble, but it can be verified by dimensional analysis. Thus a thick wire is very much harder to twist than a thin one. A wire of narrow rectangular cross-section, such as a strip or a ribbon has a relatively small torsion constant. Now let's look not at a torsion pendulum, but at a pendulum swinging about an axis under gravity. mg We suppose the pendulum, of mass m , is swinging about a point O , which is at a distance h from the centre of mass C . The rotational inertia about O is I . The line OC makes an angle θ with the

19 vertical, so that the horizontal distance between O and C is $h \sin \theta$. $mgh \sin \theta$, so that the equation of motion is For small angles, this is The torque

about O is $I\ddot{\theta} = -mgh \sin \theta$. 2.10.1 $I\ddot{\theta} = -mgh\theta$. This is simple harmonic motion of period $p = 2\pi \sqrt{I/mgh}$. 2.10.2 2.10.3 We'll look at two examples - a uniform rod, and an arc of a circle. First, a uniform rod. O The centre of mass is C. The rotational inertia about C is $ml^2/12$, so the rotational inertia about O is $I = ml^2/12 + mh^2$. If we substitute this in equation 2.10.3, we find for the period of small oscillations $p = 2\pi \sqrt{I/mgh}$. 2.10.4 This can be written

$p = 2\pi \sqrt{\frac{1}{g} \left(\frac{l^2}{12} + h^2 \right) / h}$. 2.10.5 or, if we write P and h/l 2.10.6 The figure shows a graph of P versus h . $g = 9.8 \text{ m/s}^2$ Distance h between O and C Equation 2.10.6 can be written $P^2 = \frac{4\pi^2}{g} \left(\frac{l^2}{12h} + h \right)$. 2.10.7 h and, by differentiation of P^2 with respect to h , we find that the period is least when $h = l/\sqrt{3}$. This least period is given by $P = 2\pi \sqrt{l/g}$, or $P = 1.861 \sqrt{l/g}$.

2.1 Equation 2.10.7 can also be written $3D: -P/D + 1 = 0$. 2.10.8 This quadratic equation shows that there are two positions of the support O that give rise to the same period of small oscillations. The period is least when the two solutions of equation 2.10.8 are equal, and by the theory of quadratic equations, then, the least period is given by as we also deduced by differentiation of equation 2.10.7, and this occurs when $h = l/\sqrt{3}$. For periods longer than this, there are two solutions for h . Let h_1 be the smaller of these, and let h_2 be the larger. By the theory of quadratic equations, we have $h_1 + h_2 = P^2 \cdot 3g/4\pi^2$ and $h_1 h_2 = l^2/3$. 2.10.9 2.10.10 Let $H = h_2 - h_1$ be the distance between two points O that give the same period of oscillation. Then $H^2 = (h_2 - h_1)^2 = (h_2 + h_1)^2 - 4h_1 h_2 = \left(\frac{3gP^2}{4\pi^2} \right)^2 - \frac{4l^2}{3}$. 2.10.11 9 If we measure H for a given period P and recall the definition of P we see that this provides a method for determining g . Although this is a common undergraduate laboratory exercise, the graph shows that the minimum is very shallow and consequently H and hence g are very difficult to measure with any precision. For another example, let us look at a wire bent into the arc of a circle of radius a oscillating in a vertical plane about its mid-point. In the figure, C is the centre of mass.

2.2 The rotational inertia about the centre of the circle is ma^2 . 2. axes theorem, we see that the $I = ma^2 - m(a-h)^2 + mh^2 = 2mah$. By two applications of the parallel rotational inertia about the point of oscillation is Thus, from equation 2.10.3 we find $P = 2\pi \sqrt{I/mgh} = 2\pi \sqrt{2ah/g}$. 2.10.12 and the period is independent of the length of the arc. 2.1 1. Plane Laminae. Product moment. Translation of Axes (Parallel Axes Theorem). We consider a set of point masses distributed in a plane, or a

plane lamina. We have hitherto met three second moments of inertia: 2.11.1 $I_x = \int y^2 dm$; 2.11.2 $I_y = \int x^2 dm$; 2.11.3 These are respectively the moments of inertia about the x- and y-axes (assumed to be in the plane of the masses or the lamina) and the z-axis (normal to the plane). Clearly, $I_z = I_x + I_y$, which is the perpendicular axes theorem for a plane lamina. We now introduce another quantity, H , called the product moment of inertia with respect to the x- and y-axes, defined by $H = \int xy dm$. 2.11.4 We'll need sometime to ask ourselves whether this has any particular physical significance, or whether it is merely something to calculate for the sake of passing the time of day. In the meantime, the reader should recall the parallel axes theorems (Section 2.5) and, using arguments similar to those given in that section, should derive $H = H_c + My_c$. 2.11.5 It may also be noted that equation 2.11.4 does not contain any squared terms and therefore the product moment of inertia, depending on the distribution of masses, is just as likely to be a negative quantity as a positive one.

23 We shall defer discussing the physical significance, if any, of the product moment until section 12. In the meantime let us try to calculate the product moment for a plane right triangular lamina: OAB . The area of the triangle is $\frac{1}{2}ab$ and so the mass of the element $dx dy$ is $\frac{2M}{ab} dx dy$ where M is the mass of the complete triangle. The product moment of the element with respect to the sides OA , OB is $2Mxy dx dy$ and so the product moment of the entire triangle is $2M \int \int xy dx dy$. We have to consider carefully the limits of integration. We'll integrate first with respect to x ; that is to say we integrate along the horizontal (y constant) strip from the side OB to the side AB . That is to say we integrate x from where $x = 0$ to where $x = a(1 - \frac{y}{b})$. The product moment is therefore $\frac{2M}{ab} \int_0^b y^2 \left(\frac{a^2}{2} - \frac{a^2 y}{b} + \frac{a^2 y^2}{2b^2} \right) dy$. We now have to add up all the horizontal strips from the side OA , where $y = 0$, to B , where $y = b$. Thus $\int_0^b y^2 dy$, which, after some algebra, comes to $H = \frac{Mab}{6}$.

24 The coordinates of the centre of mass with respect to the sides OA , OB are $(\frac{a}{3}, \frac{b}{3})$, so that, from equation 2.11.5, we find that the product moment with respect to axes parallel to OA , OB and passing through the centre of mass is $-\frac{Mab}{6}$. 2.12 Rotation of Axes. $Ox'y'$ Ox, y We start by recalling a result from elementary geometry. Consider two sets of axes $Ox'y'$ and Ox, y , the latter being inclined at an angle θ to the former. Any point in the plane can be described by the coordinates (x, y) or by (x', y') . matrix: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ These

coordinates are related by a rotation 2.12.1 $(x, y) = (\cos\theta - \sin\theta \begin{pmatrix} x \\ y \end{pmatrix}$, (sin0 2.12.2
The rotation matrix is orthogonal; one of the several properties of an
orthogonal matrix is that its reciprocal is its transpose. Now let us apply this
to the moments of inertia of a plane lamina. Let us suppose that the axes are
in the plane of the lamina and that O is the centre of mass of the lamina. A, B
and H are the moments of inertia with respect to the axes Oxy, and A, B and
H are the moments of inertia with respect to Oxy. Strictly speaking a lamina
implies a continuous distribution of matter in a plane, but, since matter, we
are told, is composed of discrete atoms, there is little difficulty in justifying
treating a lamina as though it we a distribution of point masses in the plane.
In any case the results that follow are valid either for a collection of point
masses in a plane or for a genuine continuous lamina.

We have, by definition: 2.12.3 $A = \sum m x^2$, $B = \sum m y^2$, 2.12.4 $H = \sum m xy$, 2.12.5
Now let us apply equation 2.12.1 to equation 2.12.3: $A = \sum m (-x \sin\theta + y \cos\theta)^2$:
 $\sin\theta \cos\theta$: $O \sum m x$: $-2 \sin\theta \cos\theta \sum m xy + \cos^2\theta \sum m y^2$. That is to say
(writing the third term first, and the first term last) $A = A \cos^2\theta - 2H \sin\theta \cos\theta + B \sin^2\theta$. 2.12.6
In a similar fashion, we obtain for the other two
moments $B = A \sin^2\theta + 2H \sin\theta \cos\theta + B \cos^2\theta$ 2.12.7 and $H = (A - B) \sin\theta \cos\theta + H(\cos^2\theta - \sin^2\theta)$ 2.12.8
It is usually more convenient to make
use of trigonometric identities to write these as $A = \frac{1}{2}(A+B) + \frac{1}{2}(A-B)\cos 2\theta - H \sin 2\theta$, 2.12.9
 $B = \frac{1}{2}(A+B) - \frac{1}{2}(A-B)\cos 2\theta + H \sin 2\theta$, 2.12.10
 $H = \frac{1}{2}(A-B)\sin 2\theta + H \cos 2\theta$. 2.12.11
These equations enable us to calculate the
moments of inertia with respect to the axes Oxy if we know the moments
with respect to the axes Oxy. Further, a matter of importance, we see, from
equation 2.12.11, that if $2H \tan 2\theta = A - B$, 2.12.12 the product moment H with
respect to the axes Oxy is zero. This gives some physical meaning to the
product moment, namely: If we can find some axes (which we can, by means
of equation 2.12.12) with respect to which the product moment is zero, these
axes are called the principal axes of the lamina, and the moments of inertia
with respect to the principal axes are called the

26 principal moments of inertia. I shall use the symbols A_0 and B_0 for the
principal moments of inertia, and I shall adopt the convention that $A_0 < B_0$.
Example: Consider three point masses at the coordinates given below: Mass
Coordinates 5 (1,1) 3 (4,2) 2 (3,4) The moments of inertia are $A = 49$, $B = 71$, $C = 53$.
The coordinates of the centre of mass are (2.3, 1.9). If we use the

parallel axes theorem, we can find the moments of inertia with respect to axes parallel to the original ones but with origin at the centre of mass. With respect to these axes we find $I_x = 12.9$, $I_y = 18.1$, $I_{xy} = +9.3$. The principal axes are therefore inclined at angles θ to the x-axis given (equation 2.13.12) by $\tan 2\theta = 3.57669$; That is $\theta = 37^\circ 11'$ and $127^\circ 11'$. On using equation 2.12.9 or 10 with these two angles, together with the convention that $-40^\circ < \theta < 40^\circ$, we obtain for the principal moments of inertia $I_{x_1} = 5.84$ and $I_{y_1} = 25.16$.

Example. Consider the right-angled triangular lamina of section 11. The moments of inertia with respect to axes passing through the centre of mass and parallel to the orthogonal sides of the triangle are $I_x = Mb^2/6$, $I_y = 8Ma^2/3$, $I_{xy} = -6Mab/5$. The angles that the principal axes make with the a - side are given by $\tan 2\theta = b^2/a^2$. The interested reader will be able to work out expressions, in terms of M, a, b, for the principal moments. 2.13 Moment of Inertia of an Ellipse

27 Consider a plane lamina such that its radius of gyration about some axis through the centre of mass is k. Let P be a vector in the direction of that axis, originating at the centre of mass, given by $\mathbf{P} = a\mathbf{u}/k$. 2.13.1 Here is a unit vector in the direction of interest; k is the radius of gyration, and a is an arbitrary length introduced so that the dimensions of P are those of length, and the length of the vector P is inversely proportional to the radius of gyration. The moment of inertia is $Mk^2 = Ma^4/p^2$. That is to say $Ma^4/p^2 = A\cos^2\theta - 2H\sin\theta\cos\theta + B\sin^2\theta$, 2.13.2 where A, H and B are the moments with respect to the x- and y-axes. Let (x, y) be the coordinates of the tip of the vector P, so that $x = P\cos\theta$ and $y = P\sin\theta$. Then $Ma^4 = Ax^2 - 2Hxy + By^2$. 2.13.3 Thus, no matter what the shape of the lamina, however irregular and asymmetric, the tip of the vector P traces out an ellipse, whose axes are inclined at angles θ to the x-axis. This is the momental ellipse, and the axes of the momental ellipse are the principal axes of the lamina. Example. Consider a regular n-gon. By symmetry the moment of inertia is the same about any two axes in the plane inclined at $2\pi/n$ to each other. This is possible only if the momental ellipse is a circle. It follows that the moment of inertia of a uniform polygonal plane lamina is the same about any axis in its plane and passing through its centroid. Exercise. Show that the moment of inertia of a uniform plane n-gon of side 2a about any axis in its plane and passing through its centroid is $ma^2(20 + 3 \cot(\pi/n))$. What is this for a square? For an equilateral triangle? 2.14. Eigenvectors and eigenvalues. In sections

11-13, we have been considering some aspects of the moments of inertia of plane laminas, and we have discussed such matters as rotation of axes, and such concepts as product moments of inertia, principal axes, principal moments of inertia and the momental ellipse. We next need to develop the same concepts with respect to three-dimensional solid bodies. In doing

28 so, we shall need to make use of the algebraic concepts of eigenvectors and eigenvalues. If you are already familiar with such matters, you may want to skip this section and move on to the next. If the ideas of eigenvalues and eigenvectors are new to you, or if you are a bit rusty with them, this section may be helpful. I do assume that the reader is at least familiar with the elementary rules of matrix multiplication. Consider what happens when you multiply a vector, for example the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, by a square matrix, for example the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. We obtain: $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$. The result of the operation is another vector that is in quite a different direction from the original one. If the matrix is merely to multiply the vector by 3 without changing its direction. The vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a very special one, and it is called an eigenvector of the matrix, and the multiplier 3 is called the corresponding eigenvalue. "Eigen" is German for "own" in the sense of "my own book". There is one other eigenvector of the matrix; it is the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Try it; you should find that the corresponding eigenvalue is 2. In short, given a square matrix A , if you can find a vector a such that $Aa = La$, where L is merely a scalar multiplier that does not change the direction of the vector a , then a is an eigenvector and L is the corresponding eigenvalue. In the above, I told you what the two eigenvectors were, and you were able to verify that they were indeed eigenvectors and you were able to find their eigenvalues by straightforward arithmetic. But, what if I hadn't told you the eigenvectors? How would you find them? A12 Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; we must have and let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigenvector with corresponding eigenvalue L . Then

That is, $Ax = Lx$, or $(A - L) x = 0$. These two equations are consistent only if the determinant of the coefficients is zero. That is, $\begin{vmatrix} a-L & b \\ c & d-L \end{vmatrix} = 0$. This equation is a quadratic equation in L , known as the characteristic equation, and its two roots, the characteristic or latent roots are the eigenvalues of the matrix. Once the eigenvalues are found the ratio of x_1 to x_2 is easily found, and hence the eigenvectors. Similarly, if A is a 3×3 matrix, the characteristic equation is $\begin{vmatrix} a-L & b & c \\ d & e-L & f \\ g & h & i-L \end{vmatrix} = 0$. This is a cubic equation in L , the three

roots being the eigenvalues. For each eigenvalue, the ratio $x : x^2 : x^3$ can easily be found and hence the eigenvectors. The characteristic equation is a cubic equation, and is best solved numerically, not by algebraic formula. The cubic equation can be written in the form $3 + a^2X + aX + a = 0$, and the solutions can be checked from the following results from the theory of equations: -%.

30 2.15. Solid body. The moments of inertia of a collection of point masses distributed in three-dimensional space (or of a solid three-dimensional body, which, after all, is a collection of point masses (atoms)) with respect to axes Oxyz are: $B = \sum m(z^2 + x^2)$ $C = \sum m(x^2 + y^2)$ $F = \sum myz$ $G = \sum mzx$ $H = \sum y'mxy$. Suppose that t , B , C , F , G , H , are the moments and products of inertia with respect to axes whose origin is at the centre of mass. The parallel axes theorems (which the reader should prove) are as follows: Let P be some point not at the centre of mass, such that the coordinates of the centre of mass with respect to axes parallel to the axes Oxyz but with origin at P are $(\bar{x}, \bar{y}, \bar{z})$. Then the moments and products of inertia with respect to the axes through P are $A[\bar{x}^2 + M(\bar{y}^2 + \bar{z}^2)] + B + M(\bar{y}^2 + \bar{z}^2)$ $B + M(\bar{x}^2 + \bar{z}^2) + F +$ where M is the total mass. Unless stated otherwise, in what follows we shall suppose that the moments and products of inertia under discussion are referred to a set of axes with the centre of mass as origin. 2.16 Rotation of axes - three dimensions. Let Oxyz be one set of mutually orthogonal axes, and let Ox₁y₁z₁ be another set of axes inclined to the first. The coordinates (x_1, y_1, z_1) of a point with respect to the second basis set are related to the coordinates (x, y, z) with respect to the first by $x_1 = c_{11}x + c_{12}y + c_{13}z$ etc. 2.16.1 Here the c_{ij} are the cosines of the angles between the axes of one basis set with respect to the axes of the other. For example, c_{11} is the cosine of the angle between Ox and Ox₁. c_{12} is the cosine of the angle between Oy and Ox₁. c_{23} is the cosine of the angles between Oy and Oz.

31 Some readers may know how to express these cosines in terms of complicated expressions involving the Eulerian angles. While these are important, they are not essential for following the present development, so we shall not make use of the Eulerian angles just here. The matrix of direction cosines is orthogonal. Among the several properties of an orthogonal matrix is the fact that its reciprocal (inverse) is equal to its transpose - i.e. the reciprocal of an orthogonal matrix is found merely by interchanging the rows and columns. This enables us easily to find (x, y, z) in terms of (x_1, y_1, z_1) . A

number of other properties of an orthogonal matrix are useful in detecting, locating and even correcting arithmetic mistakes in computing the elements. These properties are

- The sum of the squares of the elements in any row or column is unity. This merely expresses the fact that the magnitude of a unit vector along any of the six axes is indeed unity. The sum of the products of corresponding elements of any two rows or of any two columns is zero. This merely expresses the fact that the scalar product of any two orthogonal vectors is zero. It will be noted that checking for property 1 will not detect any mistakes in sign of the elements, whereas checking for property 2 will do so.
- Every element is equal to $\pm$ its own cofactor. This expresses the fact that the cross product of two unit orthogonal vectors is equal to the third. The determinant of the matrix is ± 1 . If the sign is negative, it means that the chiralities (handedness) of the two basis sets of axes are opposite; i.e. one of them is a right-handed set and the other is a left-handed set. It is usually convenient to choose both sets as right-handed. If it is possible to find a set of axes with respect to which the product moments F , G and H are all zero, these axes are called the principal axes of the body, and the moments of inertia with respect to these axes are the principal moments of inertia, for which we shall use the notation A_0 , B_0 , C_0 , with the convention $A_0 \leq B_0 \leq C_0$. We shall see shortly that it is indeed possible, and we shall show how to do it. A vector whose length is inversely proportional to the radius of gyration traces out in space an ellipsoid, known as the momental ellipsoid. In the study of solid body rotation (whether by astronomers studying the rotation of asteroids or by chemists studying the rotation of molecules) bodies are classified as follows.

- $A_0 : B_0 : C_0$ The ellipsoid is a triaxial ellipsoid, and the body is an asymmetric top.
- $A_0 < B_0 = C_0$ The ellipsoid is a prolate spheroid and the body is a prolate symmetric top.
- $A_0 = B_0 < C_0$ The ellipsoid is an oblate spheroid and the body is an oblate symmetric top.
- $A_0 = B_0 = C_0$ The ellipsoid is a sphere and the body is a spherical top.
- One moment is zero. The ellipsoid is an infinite elliptical cylinder, and the body is a linear top.

32 Example. We know from section 2.5 that the moment of inertia of a plane square lamina of side $2a$ about an axis through its centroid and perpendicular to its area is $\frac{1}{3}ma^2$, and it will hence be obvious that the moment of inertia of a uniform solid cube of side $2a$ about an axis passing through the mid-points of opposite sides is also $\frac{1}{3}ma^2$. It will clearly be the

same about an axis passing through the mid-points of any pairs of opposite sides. Therefore the cube is a spherical top and the momental ellipse is a sphere. Therefore the moment of inertia of a uniform solid cube about any axis through its centre (including, for example, a diagonal) is also $\frac{1}{12}ma^2$.

Example. What is the ratio of the length to the diameter of a uniform solid cylinder such that it is a spherical top? [Answer: I make it $x/2 = 0.866$.] Let us note in passing that $A + B + C = 2m(x^2 + y^2 + z^2) = 2mr^2$, 2.16.2 which is independent of the orientation of the basis axes. In other words, regardless of how A , B and C may depend on the orientation of the axes with respect to the body, the sum $A + B + C$ is invariant under a rotation of axes. We shall deal with the determination of the principal axes in section 2.18 - but don't skip section 2.17.

2.17 Solid Body Rotation. The Inertia Tensor. It is intended that this chapter should be limited to the calculation of the moments of inertia of bodies of various shapes, and not with the huge subject of the rotational dynamics of solid bodies, which requires a chapter on its own. In this section I mention merely for interest two small topics involving the principal axes, and a third topic in a bit more detail as necessary before proceeding to section 2.18. Everyone knows that the relation between translational kinetic energy and linear momentum is $E = p^2/(2m)$. Similarly rotational kinetic energy is related to angular momentum L by $E = L^2/(2I)$, where I is the moment of inertia. If an isolated body (such as an asteroid) is rotating about a non-principal axis, it will be subject to internal stresses. If the body is nonrigid this will result in distortions (strains) which may cause the body to vibrate. If in addition the body is inelastic the vibrations will rapidly die out (if the damping is greater than critical damping, indeed, the body will not even vibrate). Energy that was originally rotational kinetic energy will be converted to heat (which will be radiated away.) The body loses rotational kinetic energy. In the absence of external torques, however, L remains constant. Therefore, while E diminishes, I increases. The body adjusts its rotation until it is rotating around its axis of

33 maximum moment of inertia, at which time there are no further stresses, and the situation remains stable. In general the rotational motion of a solid body whose momental ellipse is triaxial is quite complicated and chaotic, with the body tumbling over and over in apparently random fashion. However, if the body is nonrigid and inelastic (as all real bodies are in practice), it will eventually end up rotating about its axis of maximum

moment of inertia. The time taken for a body, initially tumbling chaotically over and over, until it reaches its final blissful state of rotation about its axis of maximum moment of inertia, depends on how fast it is rotating. For most irregular small asteroids the time taken is comparable to or longer than the age of formation of the solar system, so that it is not surprising to find some asteroids with non-principal axis (NPA) rotation. However, a few rapidly-rotating NPA asteroids have been discovered, and, for rapid rotators, one would expect PA rotation to have been reached a long time ago. It is thought that something (such as a collision) must have happened to these rapidly-rotating NPA asteroids relatively recently in the history of the solar system. Another interesting topic is that of the stability of a rigid rotator that is rotating about a principal axis, against small perturbations from its rotational state. Although I do not prove it here (the proof can be done either mathematically, or by a qualitative argument) rotation about either of the axes of maximum or of minimum moment of inertia is stable, whereas rotation about the intermediate axis is unstable. The reader can observe this for him- or herself. Find anything that is triaxial - such as a small block of wood shaped as a rectangular parallelepiped with unequal sides. Identify the axes of greatest, least and intermediate moment of inertia. Toss the body up in the air at the same time setting it rotating about one or the other of these axes, and you will be able to see for yourself that the rotation is stable in two cases but unstable in the third. I now deal with a third topic in rather more detail, namely the relation between angular momentum $\mathbf{L}$, and angular velocity $\boldsymbol{\omega}$. The reader will be familiar from elementary (and two- dimensional) mechanics with the relation $\mathbf{L} = I\boldsymbol{\omega}$. What we are going to find in the three-dimensional solid-body case is that the relation is $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$. Here $\mathbf{L}$, and $\boldsymbol{\omega}$ are, of course, vectors, but they are not necessarily parallel to each other. They are parallel only if the body is rotating about a principal axis of rotation. The quantity $\mathbf{I}$ is a tensor known as the inertia tensor. Readers will be familiar with the equation $\mathbf{F} = m\mathbf{a}$. Here the two vectors are in the same direction, and m is a scalar quantity that does not change the direction of the vector that it multiplies. A tensor usually (unless its matrix representation is diagonal) changes the direction as well as the magnitude of the vector that it multiplies. The reader might like to think of other examples of tensors in physics. There are several. One that comes to mind is the permittivity of an anisotropic crystal; in the equation $\mathbf{D} = \boldsymbol{\epsilon}\mathbf{E}$, $\mathbf{D}$ and $\mathbf{E}$ are not parallel unless they are both directed along one of the crystallographic axes. If there are no external

torques acting on a body, $\mathbf{l}$, is constant in both magnitude and direction. The instantaneous angular velocity vector, however, is not fixed either in space or with respect to the body - unless the body is rotating about a principal axis and the inertia tensor is diagonal. So much for a preview and a qualitative description. Now down to work.

34 I am going to have to assume familiarity with the equation for the components of the cross product of two vectors: I am also going to assume that the reader knows that the angular momentum of a particle of mass m at position vector $\mathbf{r}$ (components (x,y,z)) and moving with velocity $\mathbf{v}$ (components (v_x, v_y, v_z)) is $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$. For a collection of particles, (or an extended solid body, which, I'm told, consists of a collection of particles called atoms), the angular momentum is $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$:
$$L_x = m(yv_z - zv_y) + m(zv_x - xv_z) + m(xv_y - yv_x)$$
 I also assume that the relation between linear velocity $\mathbf{v}$ (components (v_x, v_y, v_z)) and angular velocity $\boldsymbol{\omega}$ (components $(\omega_x, \omega_y, \omega_z)$) is understood to be $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, so that, for example, $v_x = \omega_y z - \omega_z y$. Then
$$L_x = \sum m(yv_z - zv_y) = \sum m(y(\omega_y z - \omega_z y) - z(\omega_x y - \omega_y x)) = \sum m(\omega_y yz - \omega_z y^2 - \omega_x yz + \omega_y xz) = \sum m(\omega_y yz - \omega_z y^2 - \omega_x yz + \omega_y xz)$$
 Finally, we obtain
$$L_x = I_{xx}\omega_x + I_{yy}\omega_y + I_{zz}\omega_z + I_{xy}\omega_y + I_{yx}\omega_x + I_{yz}\omega_z + I_{zy}\omega_y + I_{zx}\omega_x + I_{xz}\omega_z + I_{zx}\omega_x + I_{xz}\omega_z$$

35 so that, for example, $I_{xy} = -I_{yx}$. It is a symmetric matrix (but it is not an orthogonal matrix). 2.18. Determination of the Principal Axes. We now need to address ourselves to the determination of the principal axes. Unlike the two-dimensional case, we do not have a nice, simple explicit expression similar to equation 2.12.12 to calculate the orientations of the principal axes. The determination is best done through a numerical example. Consider four masses whose positions and coordinates are as follows:
$$\begin{array}{c} M \\ x \\ y \\ z \end{array} \begin{array}{cccc} 1 & 3 & 1 & 4 \\ 2 & 1 & 5 & 9 \\ 3 & 2 & 6 & 5 \\ 4 & 3 & 5 & 9 \end{array}$$
 Relative to the first particle, the coordinates are $(1, 0, 0, 0, 2, -2, 4, 5, 3, -1, 5, 1, 4, 0, 4, 5)$. From this, it is easily found that the coordinates of the centre of mass relative to the first particle are $(-0.7, 3.9, 3.3)$, and the moments of inertia with respect to axes through the first particle are $A = 324$, $B = 164$, $C = 182$, $F = 135$, $G = -23$, $H = -31$. From the parallel axes theorems we can find the moments of inertia with respect to axes passing through the centre of mass:

36 $A = 63.0$, $B = 50.2$, $C = 25.0$, $F = 6.3$, $G = 0.1$, $H = -3.7$. The inertia tensor is therefore

63.0 3.7 -0.1) -6.3 3.7 50.2 -0.1 -6.3 25.0 We understand from what has been written previously that if ω , the instantaneous angular velocity vector, is along any of the principal axes, then $I\omega$ will be in the same direction as ω . In other words, if (l, m, n) are the direction cosines of a principal axis, then $-G l$ $H B G -F = X$ where L is a scalar quantity. In other words, a vector with components l, m, n (direction cosines of a principal axis) is an eigenvector of the inertia tensor, and L is the corresponding principal moment of inertia. There will be three eigenvectors (at right angles to each other) and three corresponding eigenvalues, which we'll initially call L, L_2, L_3 , though, as soon as we know which is the largest and which the smallest, we'll call A_0, B_0, C_0 , according to our convention $A_0 \leq B_0 \leq C_0$. The characteristic equation is $L^3 - (A_0 + B_0 + C_0)L^2 + (A_0B_0 + B_0C_0 + C_0A_0)L - A_0B_0C_0 = 0$. In this case, this results in the cubic equation $a_0 + a_1L + a_2L^2 + a_3L^3 = 0$, where

$a_3 = 1, a_2 = -76226.44, a_1 = -5939.21, a_0 = 138.20$ The three solutions for L , which we shall call A_0, B_0, C_0 in order of increasing size are $A_0 = 23.498256, B_0 = 50.627521, C_0 = 64.074223$ and these are the principal moments of inertia. From the theory of equations, we note that the sum of the roots is exactly equal to $-a_2/a_3$, and we also note that it is equal to $A_0 + B_0 + C_0$, consistent with what we wrote in section 2.16. (See equation 2.16.2) The sum of the diagonal elements of a matrix is known as the trace of the matrix. Mathematically we say that "the trace of a symmetric matrix is invariant under an orthogonal transformation". Two other relations from the theory of equations may be used as a check on the correctness of the arithmetic. The product of the solutions equals $-a_0/a_3$, which is also equal to the determinant of the inertia tensor, and the sum of the products taken two at a time equals $-a_1/a_3$. We have now found the magnitudes of the principal moments of inertia; we have yet to find the direction cosines of the three principal axes. Let's start with the axis of least moment of inertia, for which the moment of inertia is $A_0 = 23.498256$. Let the direction cosines of this axis be (l, m, n) . Since this is an eigenvector with eigenvalue A_0 we must have $63.0 3.7 3.7 50.2 -0.1 -6.3 -6.3 m = 23.498256 m 25.0 n$ These are three linear equations in l, m, n , with no constant term. Because of the lack of a constant term, the theory of equations tells us that the third equation, if it is consistent with the other two, must be a linear combination of the first two. We have, in effect, only two independent equations, and we are going to need a third, independent equation if we are to solve for the three direction cosines. If we let $l = \sqrt{1 - m^2 - n^2}$ and

$m' = m/n$, then the first two equations become $39.501744l' + 3.7m' - 0.1 = 0$
 $3.7l' + 26.701744m' - 6.3 = 0$. The solutions are $l' = -0.019825485$ $m' =$
 $+0.238686617$.

38 The correctness of the arithmetic can and should be checked by verifying that these solutions also satisfy the third equation. The additional equation that we need is provided by Pythagoras's theorem, which gives for the relation between three direction cosines $l^2 + m^2 + n^2 = 1$, or $l' + m' + n' = 1$ whence $n = 0.972495608$. Thus we have, for the direction cosines of the axes corresponding to the moments of inertia A_0 , $l = 0.019280197$ $m = 0.232121881$ $n = 0.972495608$ (Check that $l^2 + m^2 + n^2 = 1$.) It does not matter which sign you choose - after all, the principal axis goes both ways. Similar calculations for B_0 yield $l = 0.280652440$ $m = 0.932312706$ $n = 0.228094774$ and for C_0 $l = 0.959615796$ $m = 0.277330987$ $n = 0.047170415$. For the first two axes, it does not matter whether you choose the upper or the lower sign. For the third axes, however, in order to ensure that the principal axes form a right-handed set, choose the sign such that the determinant of the matrix of direction cosines is $+1$.

39 We have just seen that, if we know the moments and products of inertia A, B, C, F, G, H with respect to some axes (i.e. if we know the elements of the inertia tensor) we can find the principal moments of inertia A_0, B_0, C_0 by diagonalizing the inertia tensor, or finding its eigenvalues. If, on the other hand, we know the principal moments of inertia of a system of particles (or of a solid body, which is a collection of particles), how can we find the moment of inertia I about an axis whose direction cosines with respect to the principal axes are (l, m, n) ? First, some geometry. Let $Oxyz$ be a coordinate system, and let $P(x, y, z)$ be a point whose position vector is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Let L be a straight line passing through the origin, and let the direction cosines of this line be (l, m, n) . A unit vector $\mathbf{e}$ directed along L is represented by $\mathbf{e} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$. The angle θ between $\mathbf{r}$ and $\mathbf{e}$ is found from the scalar product $\mathbf{r} \cdot \mathbf{e}$, given by $\cos \theta = \frac{\mathbf{r} \cdot \mathbf{e}}{r} = \frac{lx + my + nz}{r}$. The perpendicular distance p from P to L is $p = r \sin \theta = \sqrt{r^2 - (\mathbf{r} \cdot \mathbf{e})^2}$. If we write $\sin^2 \theta = 1 - \cos^2 \theta$, we soon obtain $p^2 = (x^2 + y^2 + z^2) - (lx + my + nz)^2$. Now return to our collection of particles, and let $Oxyz$ be the principal axes of the system. The moment of inertia of the system with respect to the line L is $I = \sum m_i p_i^2 = \sum m_i (x_i^2 + y_i^2 + z_i^2 - (l x_i + m y_i + n z_i)^2)$. Noting that

$I_x^2 = I - m^2 - n^2$, $m^2 = I - n^2 - I_x^2$, $n^2 = I - I_x^2 - m^2$, we find, after further manipulation:

40 where I have omitted a subscript i on each symbol. Making use of the expression for p and noting that the product moments of the system with respect to $Oxyz$ are all zero, we obtain $I_x^2 = I - m^2 - n^2$. 2.18.1 Also, let A, B, C, F, G, H be the moments and products of inertia with respect to a set of nonprincipal orthogonal axes; then the moment of inertia about some other axis with direction cosines l, m, n with respect to these nonprincipal axes is $I = I_x A + m^2 B + n^2 C - 2lmF - 2nlG - 2mlH$. 2.18.2 Example. A Brick. We saw in section 16 that the moment of inertia of a uniform solid cube of mass M and side $2a$ about a body diagonal is $\frac{1}{2} Ma^2$, and we saw how very easy this was. At that time the problem of finding the moment of inertia of a uniform solid rectangular parallelepiped of sides $2a, 2b, 2c$ must have seemed intractable, but by now it is not at all hard. Thus we have: $A = M(b^2 + c^2)$, $B = M(c^2 + a^2)$, $C = M(a^2 + b^2)$, $F = abc$, $G = abc$, $H = abc$.

41 We obtain: $I = 2M(b^2 + c^2 + a^2 + b^2 + c^2 + a^2) = 3M(a^2 + b^2 + c^2)$. We note: i. ii. 111. This is dimensionally correct; It is symmetric in a, b, c ; If $a = b = c$, it reduces to $\frac{1}{2} Ma^2$. 2.19 Moment of Inertia with Respect to a Point. By "moment of inertia" we have hitherto meant the second moment of mass with respect to an axis. We were easily able to identify it with the rotational inertia with respect to the axis, namely the ratio of an applied torque to the resulting angular acceleration. I am now going to define the (second) moment of inertia with respect to a point, which I shall take unless otherwise specified to mean the origin of coordinates. If we have a collection of mass points m_i at distances r_i from the origin, I define I : $I = \sum m_i r_i^2 = \sum m_i (x_i^2 + y_i^2 + z_i^2)$. 2.19.1 I is the (second) moment of inertia with respect to the origin, also sometimes called the "geometric moment of inertia". I cannot relate it in an obvious way to a simple dynamical concept in the same way that I related moment of inertia with respect to an axis to rotational inertia, but we shall see that it is by no means merely a tedious exercise in arithmetic, and it does have its uses. The symbol I has probably been used rather a lot in this chapter; so to describe the geometric moment of inertia I am going to use the symbol I not in italics. The moment of inertia with respect to the origin is clearly something that does not depend on the orientation of any particular basis set of orthogonal axes, since

it depends only on the distances of the particles from the origin. If you recall the definitions of A , B and C from section 2.15, you will easily see that $I = (A+B+C)$. 2.19.2 and we already noted (see equation 2.16.2) that $A + B + C$ is invariant under rotation of axes. In section 2.18 we expressed it slightly more generally by saying "the trace of a symmetric matrix is invariant under an orthogonal transformation". By now it probably seems slightly less mysterious.

42 Let us now calculate the geometric moment of inertia of a uniform solid sphere of radius a , mass m , density ρ with respect to the centre of the sphere. It is $I = \int r^2 dm$. sphere 2.19.3 The element of mass, dm , here is the mass of a shell of radii r , $r + dr$; that is $4\pi r^2 dr$. Thus $a \int_0^a 4\pi r^2 \rho dr$. 2.19.4 $4\pi \rho \int_0^a r^3 dr$ With $m = \frac{4}{3}\pi a^3 \rho$, this becomes $\frac{3}{2} I = \frac{1}{2} m a^2$. 2.19.5 Indeed, for any spherically symmetric distribution of matter, since $A = B = C$, it will be clear from equation 2.19.2, that the moment of inertia with respect to the centre is $\frac{3}{2}$ times the moment of inertia with respect to an axis through the centre. For example, it is obvious from the definition of moment of inertia with respect to the centre that for a hollow spherical shell it is just $\frac{2}{5} M a^2$, and therefore the moment of inertia with respect to an axis through the centre is $\frac{2}{3} M a^2$. In other words, you can work out that the moment of inertia of a hollow spherical shell with respect to an axis through its centre is $\frac{2}{3} M a^2$ in your head without any of the integration that we did in section 2.7! By way of illustration, consider three spheres, each of radius a and mass M , but the density between centre and surface varies as $P = P_0 (1 - \frac{r}{a})$, $P = P_0 (1 - \frac{r^2}{a^2})$, $P = P_0 (1 - \frac{r^3}{a^3})$ for the three spheres. Calculate for each the moment of inertia about an axis through the centre of the sphere. Express the answer in the form $\frac{1}{2} M a^2 f(k)$. Solution. The mass of a sphere is $M = 4\pi \int_0^a p(r) r^2 dr = \frac{4}{3} \pi a^3 p_0$ and so $\frac{1}{2} M a^2 \int_0^1 p(r) r^2 dr$. 5

43 The moment of inertia about the centre is $I = 4\pi \int_0^a p(r) r^4 dr$ and so the moment of inertia about an axis through the centre is $\frac{2}{3} \int_0^a p(r) r^4 dr$. $I = \frac{2}{3} \int_0^a p(r) r^4 dr$ Therefore $\frac{1}{2} I = \frac{1}{3} \int_0^a p(r) r^4 dr$ $\frac{1}{2} I = \frac{1}{3} \int_0^a p(r) r^4 dr$ For the first two spheres the integrations are straightforward. I make it for the first sphere, and for the second sphere. make the answer $I = \frac{1}{2} M a^2 (12 - 10k)$ $I = \frac{1}{2} M a^2 (12 - 9k)$ $I = \frac{1}{2} M a^2 (35 - 25k)$ $I = \frac{1}{2} M a^2 (35 - 21k)$ The integrations for the third sphere need a little more patience, but $I = \frac{1}{2} M a^2 (5(12c - 3\sin 2c - 3\sin 4c + \sin 6c) - 18\sin^2 \frac{c}{2} (4tz - \sin 4tz))$ where $\sin c = x'$. This should be enough to convince that the concept of I is

useful - but it is not its only use. We shall meet it again in Chapter 3 on the dynamics of systems of particles; in particular, it will play a role in what we shall become familiar with as the virial theorem.

2.20 Ellipses and Ellipsoids
Here are some problems concerning ellipses and ellipsoids that might be of interest. 1. Determine the principal moments of inertia of the following:

111. 44 A uniform plane lamina of mass m in the form of an ellipse of semi axes a and b . A uniform plane ring of mass m in the form of an ellipse of semi axes a and b . A uniform solid triaxial ellipsoid of mass m and semi axes a , b and c . A uniform hollow triaxial ellipsoid of mass m and semi axes a , b and c . By integration these are slightly difficult, but by physical insight they are very easy! 2. i. A uniform plane lamina is an ellipse of eccentricity e . What is the eccentricity of its momental ellipse? ii. A uniform plane ring is an ellipse of eccentricity e . What is the eccentricity of its momental ellipse? iii. The ratios of the semi axes of a uniform solid triaxial ellipsoid are $a : b : c$. What are the ratios of the semi axes of the corresponding momental ellipsoid? iv. The ratios of the semi axes of a uniform hollow triaxial ellipsoid are $a : b : c$. What are the ratios of the semi axes of the corresponding momental ellipsoid? I make the answers as follows. Let me know if you disagree. 1. i. $A = \frac{3}{2}mb^2$ $B = \frac{7}{2}ma^2$ $C = \frac{3}{2}m(a^2 + b^2)$ ii. $A = \frac{3}{2}mb^2$ $B = \frac{7}{2}ma^2$ $C = \frac{3}{2}m(a^2 + b^2)$ iii. $A = \frac{3}{2}m(b^2 + c^2)$ $B = \frac{3}{2}m(c^2 + a^2)$ $C = \frac{3}{2}m(a^2 + b^2)$ iv. $A = \frac{3}{2}m(b^2 + c^2)$ $B = \frac{3}{2}m(c^2 + a^2)$ $C = \frac{3}{2}m(a^2 + b^2)$ 2. i and ii. e . In other words, for both the lamina and the ring, the momental ellipse is the same shape as the original body. iii and iv. The same is not true, however, of the three-dimensional figures. For both the solid and the hollow hemisphere, I make the axial ratios of the corresponding momental ellipsoid $1 : a : b : c$ For example, if the axial ratios of the original ellipsoid (whether solid or hollow) are $1 : 2 : 3$, the axial ratios of the corresponding momental ellipsoid is $1 : \frac{2}{5} : \frac{3}{5} = 1 : 1.140 : 1.612$.

45 In other words, it looks as though the momental ellipsoid of an ellipsoidal body is more nearly spherical than the body itself, whereas for plane elliptical figures the body and the momental ellipse are the same shape.

CHAPTER 3 SYSTEMS OF PARTICLES 3.1 Introduction By systems of particles I mean such things as a swarm of bees, a star cluster, a cloud of gas, an atom, a brick. A brick is indeed composed of a system of particles - atoms

- which are constrained so that there is very little motion (apart from small amplitude vibrations) of the particles relative to each other. In a system of particles there may be very little or no interaction between the particles (as in a loose association of stars separated from each other by large distances) or there may be (as in the brick) strong forces between the particles. Most (perhaps all) of the results to be derived in this chapter for a system of particles apply equally to an apparently solid body such as a brick. Even if scientists are wrong and a brick is not composed of atoms but is a genuine continuous solid, we can in our imagination suppose the brick to be made up of an infinite number of infinitesimal mass and volume elements, and the same results will apply. What sort of properties shall we be discussing? Perhaps the simplest one is this: The total linear momentum of a system of particles is equal to the total mass times the velocity of the center of mass. This is true, and it may be "obvious" - but it still requires proof. It may be equally "obvious" to some that "the total kinetic energy of a system of particles is equal to $\frac{1}{2} M v^2$, where M is the total mass and v is the velocity of the center of mass" - but this one, however "obvious", is not true! Before we get round to properties of systems of particles, I want to clarify what I mean by the moment of a vector such as a force or momentum. You are already familiar, from Chapters 1 and 2, with the moments of mass, which is a scalar quantity.

3.2 Moment of a Force

First, let's look at a familiar two-dimensional situation. In figure III. 1 I draw a force F and a point O . The moment of the force with respect to O can be defined as Force times perpendicular distance from O to the line of action of F . , , FIGURE III. 1 $r \sin \theta$,,,, 0

Alternatively, (figure 111.2) the moment can be defined equally well by Transverse component of force times distance from O to the point of application of the force. $F \cos \theta = F \sin \theta$ _ FIGURE III.2 Either way, the magnitude of the moment of the force, also known as the torque, is $rF \sin \theta$. We can regard it as a vector, , perpendicular to the plane of the paper: $= r \times F$.

3.2.1

Now let me ask a question. Is it correct to say the moment of a force with respect to (or "about") a point or with respect to (or "about") an axis? In the above two-dimensional example, it does not matter, but now let me move on to three dimensions, and I shall try to clarify. In figure III.3, I draw a set of rectangular axes, and a force F , whose position vector with respect to the origin is r . F FIGURE III.3

The moment, or torque, of $\mathbf{F}$ with respect to the origin is the vector $\mathbf{z} = \mathbf{r} \times \mathbf{F}$.

3.2.2 The x-, y- and z-components of $\mathbf{z}$ are the moments of $\mathbf{F}$ with respect to the x-, y- and z- axes. You can easily find the components of $\mathbf{z}$ by expanding the cross product

3.2.2: $\mathbf{z} = y(\mathbf{F}_z - x\mathbf{F}_y) - y(\mathbf{F}_x - z\mathbf{F}_y) + z(\mathbf{F}_y - x\mathbf{F}_z)$, 3.2.3 where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors along the x, y, z axes. In figure 11.4, we are looking down the x-axis, and I have drawn the components F_y and F_z , and you can see that,

indeed, $z_x = yF_z - zF_y$. $z_y = xF_z - zF_x$ $z_z = yF_x - xF_y$ FIGURE III.4 The dimensions of moment of a force, or torque, are ML^2T^{-2} , and the SI units are N m. (It is best to leave the units as N m rather than to express torque in joules.)

3.3 Moment of Momentum In a similar way, if a particle at position $\mathbf{r}$ has linear momentum $\mathbf{p} = m\mathbf{v}$, its moment of momentum with respect to the origin is the vector $\mathbf{l}$ defined by $\mathbf{l} = \mathbf{r} \times \mathbf{p}$, 3.3.1

and its components are the moments of momentum with respect to the axes.

Moment of momentum plays a role in rotational motion analogous to the role played by linear momentum in linear motion, and is also called angular momentum. The dimensions of angular momentum are ML^2T^{-1} . Several choices for expressing angular momentum in SI units are possible; the usual choice is J s (joule seconds).

3.4 Notation I am going to establish the following notation for the purposes of this chapter, with respect to a system of particles. I shall suppose that we have n particles, and that the mass and position vector with respect to some origin of the i th particle are m_i and $\mathbf{r}_i$.

The total mass is $M = \sum m_i$, where the sum is understood to be over all particles - that is, i goes from 1 to n . A given particle may have an external force $\mathbf{F}_i$ acting upon it. (It may, of course, have several external forces acting on it, but I mean by $\mathbf{F}_i$ the vector sum of all the external forces acting on the i th particle.) It may also interact with the other particles in the system, and consequently it may have internal forces $\mathbf{F}_{ij}$ acting upon it, where j goes from 1 to n except for i . I define the vector sum $\mathbf{F} = \sum \mathbf{F}_i$ as the total external force acting upon the system. The external torque with respect to the origin of the external force $\mathbf{F}_i$ on the i th particle is $\mathbf{z}_i = \mathbf{r}_i \times \mathbf{F}_i$. I establish the following notation: Total external torque on the system with respect to the origin: 3.4.1

The linear momentum of the i th particle is $\mathbf{p}_i = m_i \mathbf{v}_i$ and the total linear momentum of the system is $\mathbf{P} = \sum \mathbf{p}_i = \sum m_i \mathbf{v}_i$. 3.4.2 The total angular momentum of the system is $\mathbf{L} = \sum \mathbf{z}_i + \sum \mathbf{r}_i \times m_i \mathbf{v}_i$. 3.4.3 The total kinetic energy of the system is $K = \frac{1}{2} \sum m_i v_i^2$. 3.4.4

(We are dealing in this chapter with "particles", whose kinetic energy is entirely translational, and does not include kinetic energy of rotation or of vibration. The symbol T , rather than K , is traditionally used for kinetic energy in advanced works on mechanics. I am not sure why this is; perhaps it is because U or V are used for potential energy. By all means use K if you prefer.) Position vector of centre of mass (see Chapter 1, especially equation 1.1.3): $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. 3.4.5 $\bar{M}$ Here the bar denotes centre of mass, and the "hats" denote unit vectors. For the velocity of the centre of mass I may use either $\mathbf{V}$ or $\mathbf{v}$. O is an arbitrary origin of coordinates. C is the centre of mass. For position vectors, unprimed single-subscript symbols will refer to O . Primed single-subscript symbols will refer to C . This will be clear, I hope, from figure III.5, which will also explain the symbols with two subscripts. In the figure I have drawn just two of the n particles - the i th and the j th. $m_i \mathbf{y} \times \mathbf{F}_i$ FIGURE III.5 Note that $\mathbf{r}_i = \mathbf{F} + \mathbf{r}$ 3.4.6

and therefore $\mathbf{r}_i = \mathbf{F} + \mathbf{r}$; 3.4.7 that is to say $\mathbf{v} = \mathbf{V} + \mathbf{v}'$. 3.4.8 Note also that $\mathbf{m} \cdot \mathbf{r} = 0$. 3.4.9 Note further that $\mathbf{m} \cdot \mathbf{v}_i = \mathbf{m}(\mathbf{v}_i - \mathbf{V})$: $\mathbf{m} \cdot \mathbf{v}_i - \mathbf{V} \cdot \mathbf{m}$: $\mathbf{M} \cdot \mathbf{V} - \mathbf{V} \cdot \mathbf{M}$: O . 3.4.10 That is, the total linear momentum with respect to the centre of mass is zero. Having established our notation, we now move on to some theorems concerning systems of particles. It may be more useful for you to conjure up a physical picture in your mind what the following theorems mean than to memorize the details of the derivations. 3.5 Linear Momentum Theorem: The total momentum of a system of particles equals the total mass times the velocity of the centre of mass. Thus: $\mathbf{P} = \sum m_i \mathbf{v}_i = \sum m_i (\mathbf{V} + \mathbf{v}_i')$: $\mathbf{M} + 0$. 3.5.1 3.6 Force and Rate of Change of Momentum Theorem: The rate of change of the total momentum of a system of particles is equal to the sum of the external forces on the system. Thus, consider a single particle. By Newton's second law of motion, the rate of change of momentum of the particle is equal to the sum of the forces acting upon it: $\frac{d\mathbf{p}_i}{dt} = \mathbf{F}_i$. 3.6.1 $\mathbf{J}$ Now sum over all the particles: $\sum \frac{d\mathbf{p}_i}{dt} = \sum \mathbf{F}_i$

3.6.2 But, by Newton's third law of motion, $\mathbf{F}_{si} + \mathbf{F}_{is} = 0$, so the theorem is proved. Corollary: If the sum of the external forces on a system is zero, the linear momentum is constant. (Law of Conservation of Linear Momentum.) 3.7 Angular Momentum Notation: $\mathbf{L}_c$ = angular momentum of system with respect to centre of mass C . $\mathbf{L}$ = angular momentum of system relative to some other origin O . $\mathbf{F}$ = position vector of C with respect to O . $\mathbf{P}$ = linear

momentum of system with respect to O. (The linear momentum with respect to C is, of course, zero.) Theorem: $L = L_c + F \times P$. 3.7.1 Thus: $L = (F \times V)_{cm} + F \times \sum m_i v_i + \sum m_i r_i \times v_i = M(F \times V) + F \times 0 + 0 \times V + L_c \therefore L = L_c + F \times P$. Example. A hoop of radius a rolling along the ground (figure 111.6): $0 = F \times P = M \times$

The angular momentum with respect to C is $L_c = I_c \omega$, where I_c is the rotational inertia about C. The angular momentum about O is therefore $L = I_c \omega + Mva = I_c \omega + Ma^2 \omega = (I_c + Ma^2) \omega = I_o \omega$, where $I = I_c + Ma^2$ is the rotational inertia about O. 3.8 Torque Notation: $Z_c =$ vector sum of all the torques about C. $z =$ vector sum of all the torques about the origin O. $F =$ vector sum of all the external forces. Theorem: $z = z_c + P \times F$. Thus: $z = \sum r_i \times F_i = \sum (r_i' + F) \times m_i v_i = \sum r_i' \times m_i v_i + F \times \sum m_i v_i \therefore z = z_c + F \times P$. 3.8.1 3.9 Comparison At this stage I compare some somewhat similar formulas. $L = L_c + F \times P$ $L = \sum m_i r_i \times v_i$ $L_c = \sum m_i r_i' \times v_i$ $P = \sum m_i r_i$ $F = \sum F_i$ $z = \sum r_i \times F_i$ $z_c = \sum r_i' \times F_i$ Notation: 3.10 Kinetic energy We remind ourselves that we are discussing particles, and that all kinetic energy is translational kinetic energy. $T_c =$ kinetic with respect to the centre of mass C. $T =$ kinetic with respect to the origin O. Theorem: $T = T_c + \frac{1}{2} M v^2$. 3.10.1

Thus: Corollary: Corollary: $T = \frac{1}{2} \sum m_i v_i^2 = \frac{1}{2} \sum m_i (v_i' + V)^2 = \frac{1}{2} \sum m_i v_i'^2 + \sum m_i v_i' \cdot V + \frac{1}{2} M V^2$. $T_c = \frac{1}{2} \sum m_i v_i'^2$. If $V = 0$, $T = T_c$. (Think about what this means.) For a non-rotating rigid body, $T_c = 0$, and therefore (Think about what this means.) 3.11 Torque and Rate of Change of Angular Momentum Theorem: The rate of change of the total angular momentum of a system of particles is equal to the sum of the external torques on the system. Thus: $\dot{L} = \sum r_i \times F_i$. 3.11.1 $\therefore L = \sum r_i \times p_i + \sum r_i \times 0$. 3.11.2 $\dot{L} = \sum \dot{r}_i \times p_i + \sum r_i \times \dot{p}_i$ But the first term is zero, because $\dot{r}_i$ and p_i are parallel. Also $0 = F + \sum F_i$. 3.11.3 $\dot{L} = \sum r_i \times \dot{p}_i$ But $\sum F_i = 0$ by Newton's third law of motion, and so $\sum r_i \times F_i$ is also zero. $\dot{L} = \sum r_i \times \dot{p}_i$ Also, $r_i \times F_i = z_i$, and so we arrive at 3.11.4 which was to be demonstrated.

10 Corollary: If the sum of the external torques on a system is zero, the angular momentum is constant. (Law of Conservation of Angular Momentum.) 3.12 Torque, Angular Momentum and a Moving Point C m_i o FIGURE III.7 In figure III.7 I draw the particle m_i , which is just one of n particles, $n - 1$ of which I haven't drawn and are scattered around in 3-space. I draw an arbitrary origin O, the centre of mass C of the system, and another

point Q, which may (or may not) be moving with respect to O. The question I am going to ask is: Does the equation $L = z$ apply to the point Q? It obviously does if Q is stationary, just as it applies to O. But what if Q is moving? If it does not apply, just what is the appropriate relation? The theorem that we shall prove - and interpret - is $L_Q = z_Q + M \mathbf{r} \times \mathbf{i}$. 3.12.1 We start: $L_Q = -(\mathbf{r}_i - \mathbf{r}_Q) \times [m_i(\mathbf{v}_i - \mathbf{V}_Q)]$. 3.12.2 $\therefore L_Q = \sum (\mathbf{r}_i - \mathbf{r}_Q) \times m_i \mathbf{v}_i + Y'(\mathbf{i}' - \mathbf{i}'_Q) \times m_i (\mathbf{v}_i - \mathbf{V}_Q)$. 3.12.3 The second term is zero, because $\mathbf{i}' = \mathbf{v}$.

11 Continue: $L_Q = -(\mathbf{r} - \mathbf{r}_Q) \times m \mathbf{v}_i - \mathbf{r}_Q \times m \mathbf{v}_i + \mathbf{r}_Q \times m \mathbf{v}$. Now $m \mathbf{v} = \mathbf{F}$, so that the first term is just Z_Q . Continue: 3.12.4 $\therefore$ Thus in general, Q.E.D. 3.12.5 $\mathbf{r} = z$, but $L_Q = z$ under any of the following three circumstances: i. $\mathbf{r} = 0$ - that is, Q coincides with C. ii. $\mathbf{i} = 0$ - that is, Q is not accelerating. iii. $\mathbf{i}$ and $\mathbf{r}$ are parallel, which would happen, for example, if O were a centre of attraction or repulsion and Q were accelerating towards or away from O. 3.13 The Virial Theorem First, let me say that I am not sure how this theorem got its name, other than that my Latin dictionary tells me that *vis*, *viris* means force, and its plural form, *vires*, *virium* is generally translated as strength. The term was apparently introduced by Rudolph Clausius of thermodynamics fame. We do not use the word strength in any particular technical sense in classical mechanics. We use the word energy to mean the ability to do work; perhaps we could use the word strength to mean the ability to exert a force. But enough of these idle speculations. Before proceeding, I define the quantity $I = \sum m_i r_i^2$ 3.13.1 as the second moment of mass of a system of particles with respect to the origin. As discussed in Chapter 2, mass is (apart from some niceties in general relativity)

12 synonymous with inertia, and the second moment of mass is used so often that it is nearly always called simply "the" moment of inertia, as though there were only one moment, the second, worth considering. Note carefully, however, that you are probably much more used to thinking about the moment of inertia with respect to an axis rather than with respect to a point. This distinction is discussed in Chapter 2, section 19. Note also that, since the symbol I tends to be heavily used in any discussion of moments of inertia, for moment of inertia with respect to a point I am using the symbol I not in italics. I can also write equation 3.13.1 as $I = \sum m_i (\mathbf{r}_i \cdot \mathbf{r}_i)$ 3.13.2 Differentiate twice with respect to time: $\ddot{I} = 2 \sum m_i (\mathbf{r}_i \cdot \ddot{\mathbf{r}}_i)$, 3.13.3 $\mathbf{i}$ and $\ddot{\mathbf{r}} = -\mathbf{m}(\ddot{\mathbf{r}} + \mathbf{r})$ 3.13.4 $\mathbf{i}$ or $4T + 2 \sum \mathbf{r}_i \cdot m_i \mathbf{i}$, 3.13.5 $\mathbf{i}$ where T is the kinetic energy of the system of

particles. The sums are understood to be over all particles - i.e. i from 1 to n . $\mathbf{F}_{ij}$ is the force on the i th particle. I am now going to suppose that there are no external forces on any of the particles in the system, but the particles interact with each other with conservative forces, $\mathbf{F}_{ij}$ being the force exerted on particle i by particle j . I am also going to introduce the notation $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$, which is a vector directed from particle i to particle j . The relation between these three vectors is shown in figure III.8. Origin

13 I have not drawn the force F_{ij} , but it will be in the opposite direction to r_{ji} if it is a repulsive force and in the same direction as r_{fi} if it is an attractive force. The total force on particle i is $\sum_j F_{ij}$, and this is equal to $\sum_j m_i a_{ij}$ 3.13.5 becomes Therefore, equation 3.13.6 is $\sum_j F_{ij} = m_i a_i$ Now it is clear that $\sum_j F_{ij} = -\sum_j F_{ji}$ 3.13.7 However, in case, like me, you find double subscripts and summations confusing and you have really no idea what equation 3.13.7 means, and it by no means at all clear, I write it out in full in the case where there are five particles. Thus: $\sum_j (F_{1j} - F_{1j} - F_{1j} - F_{1j}) - \sum_j (F_{2j} - F_{2j} - F_{2j} - F_{2j}) - \sum_j (F_{3j} - F_{3j} - F_{3j} - F_{3j}) - \sum_j (F_{4j} - F_{4j} - F_{4j} - F_{4j}) - \sum_j (F_{5j} - F_{5j} - F_{5j} - F_{5j})$. Now apply Newton's third law of motion: $\sum_j (F_{1j} - F_{1j} - F_{1j} - F_{1j}) - \sum_j (F_{2j} - F_{2j} - F_{2j} - F_{2j}) - \sum_j (F_{3j} - F_{3j} - F_{3j} - F_{3j}) - \sum_j (F_{4j} - F_{4j} - F_{4j} - F_{4j}) - \sum_j (F_{5j} - F_{5j} - F_{5j} - F_{5j})$. Now bear in mind that $\sum_j F_{ij} = -\sum_j F_{ji}$, and we see that this becomes

$14 = F_{11} + F_{12} + F_{13} + F_{14} + F_{21} + F_{22} + F_{23} + F_{24} + F_{31} + F_{32} + F_{33} + F_{34} + F_{41} + F_{42} + F_{43} + F_{44}$ and we have arrived at equation 3.13.7. Equation 3.13.6 then becomes

$$\frac{1}{2} \sum_i \sum_j \mathbf{r}_i \cdot \mathbf{F}_{ij} = - \sum_i \sum_j \frac{G m_i m_j}{r_{ij}} \quad 3.13.8$$
 I now suppose that the forces between the particles are gravitational forces, such that $\mathbf{F}_{ij} = -G m_i m_j \frac{\mathbf{r}_{ij}}{r_{ij}^3}$. 3.13.9 Now $G m_i m_j \frac{\mathbf{r}_i \cdot \mathbf{r}_{ij}}{r_{ij}^3} = -G m_i m_j \frac{r_i \cdot \mathbf{r}_{ij}}{r_{ij}^3}$ 3.13.10 which is the mutual potential energy of particles i and j . In the summation of equation 3.13.6, each pair has been counted once, and so the double summation is the total gravitational potential energy of the system of particles. Thus equation 3.13.6 becomes $\frac{1}{2} \sum_i \sum_j \mathbf{r}_i \cdot \mathbf{F}_{ij} = -2U$, 3.13.11 where T and U are the kinetic and potential energies of the system. This is one statement of the virial theorem for a system of gravitating particles. Of course, as the individual particles move around in the system, T and U are all changing from moment to moment, but always in such a manner that equation 3.13.11 is satisfied. In a stable, bound system, by which I mean that, over a long period of time, there is no long-term change in the moment of inertia of the

system, and the system is neither irreversibly dispersing or contracting, that is to say in a system in which the average value of over a long period of time is zero (I'll define "long" soon), the virial theorem for a stable, bound system of gravitating particles takes the form $2\langle T \rangle + \langle U \rangle = 0$. 3.13.12

15 Here the angular brackets are understood to mean the average values of the kinetic and potential energies over a long period of time. By a "long" period we mean, for example, long compared with the time that a particle takes to cross from one side of the system to the other, or long compared with the time that a particle takes to move in an orbit around the centre of mass of the system. (In the absence of external forces, of course, the centre of mass does not move, or it moves with a constant velocity.) For example, if a bound cluster of stars occupies a spherical volume of uniform density, $\frac{3}{5}GM^2$ the potential energy is (see equation 5.9.1 of Celestial Mechanics), so the virial theorem (equation 3.13.7) will enable you to work out the mean kinetic energy and hence speed of the stars. A globular cluster has roughly spherical symmetry, but it is not of uniform density, being centrally condensed. If you assume some functional form for the density distribution, this will give a slightly different formula for the potential energy, and you can then still use the virial theorem to calculate the mean kinetic energy. A trivial example is to consider a planet of mass m moving in a circular orbit of radius a around a Sun of mass M , such that $m \ll M$ and the Sun does not move. The potential energy of the system is $U = -GMm/a$. The speed of the planet is given by equating $-GMm/a$ to $\frac{1}{2}mv^2$, from which $T = GMm/(2a)$, so we easily see in this case that $2T + U = 0$. Problem. Show that if the force between the particles is an attractive force proportional to the n th power of the distance between the particles, the virial theorem for a stable, bound system takes the form $2\langle T \rangle - (n + 1) \langle S \rangle = 0$. For gravitating particles, $n = -2$, so that, in that case, this equation reduces to equation 3.13.12. To make a start with this problem, you can start with equation 3.13.8. Assume $r \gg r_0$ and work out what is the mutual potential energy of two particles a distance r apart.

CHAPTER 4 RIGID BODY ROTATION 4.1 Introduction No real solid body is perfectly rigid. A rotating nonrigid body will be distorted by centrifugal force* or by interactions with other bodies. Nevertheless most people will allow that in practice some solids are fairly rigid, are rotating at only a

modest speed, and any distortion is small compared with the overall size of the body. No excuses, therefore, are needed or offered for analysing to begin with the rotation of a rigid body. *! do not in this chapter delve deeply into whether there really is "such thing" as "centrifugal force". Some would try to persuade us that there is no such thing. But is there "such thing" as a "gravitational force"? And is one any more or less "real" than the other? These are deep questions best left to the philosophers. In physics we use the concept of "force" - or indeed any other concept - according as to whether it enables us to supply a description of how physical bodies behave. Many of us would, I think, be challenged if we were faced with an examination question: "Explain, without using the term centrifugal force, why Earth bulges at its equator." We have already discussed some aspects of solid body rotation in Chapter 2 on Moment of Inertia, and indeed the present Chapter 4 should not be plunged into without a good understanding of what is meant by "moment of inertia". One of the things that we found was that, while the comfortable relation $L = I\omega$ to which we are familiar with from elementary physics is adequate for problems in two dimensions, in three dimensions the relation becomes $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$, where $\mathbf{I}$ is the inertia tensor, whose properties were discussed at some length in Chapter 2. We also learned in Chapter 2 about the concepts of principal moments of inertia, and we introduced the notion that, unless a body is rotating about one of its principal axes, the equation $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$ implies that the angular momentum and angular velocity vectors are not in the same direction. We shall discuss this in more detail in this chapter. A full treatment of the rotation of an asymmetric top (whose three principal moments of inertia are unequal and which has as its momental ellipsoid a triaxial ellipsoid) is very lengthy, since there are so many cases to consider. I shall restrict consideration of the motion of an asymmetric top to a qualitative argument that shows that rotation about the principal axis of greatest moment of inertia or about the axis of least moment of inertia is stable, whereas rotation about the intermediate axis is unstable. I shall treat in more detail the free rotation of a symmetric top (which has two equal principal moments of inertia) and we shall see how it is that the angular velocity vector precesses while the angular momentum vector (in the absence of external torques) remains fixed in magnitude and direction. I shall also discuss the situation in which a symmetric top is subjected to an external torque (in which case $\mathbf{L}$ is certainly not fixed), such as the motion of a top. A similar situation, in which Earth is subject to external torques from the Sun and Moon, causes Earth's

axis to precess with a period of 26,000 years, and this will be dealt with in a chapter of the notes on Celestial Mechanics.

Before discussing these particular problems, there are a few preparatory topics, namely, angular velocity and Eulerian angles, kinetic energy, Lagrange's equations of motion, and Euler's equations of motion. 4.2 Angular Velocity and Eulerian Angles

FIGURE IV. 1 Let $Oxyz$ be a set of space-fixed axis, and let $Ox_0y_0z_0$ be the body-fixed principal axes of a rigid body. At some instant let the Eulerian angles of the set $Ox_0y_0z_0$ with respect to $Oxyz$ be (θ, ϕ, ψ) (see figure IV. 1). Those who are not familiar with Eulerian angles, which describe the orientation of one set of axes in three-dimensional space with respect to another set, should consult Chapter 3 of my notes Celestial Mechanics on this Web site. Suppose that the body is rotating about some arbitrary axis (not necessarily a principal axis), and that the instantaneous components of its angular velocity vector with respect to the principal axes are ω_1, ω_2 and ω_3 . As the body rotates, the Eulerian angles of its principal axes will be changing at rates that we shall call $\dot{\theta}, \dot{\phi}$ and $\dot{\psi}$. We are going to find how ω_1, ω_2 and ω_3 are related to $\dot{\theta}, \dot{\phi}$ and $\dot{\psi}$.

I have indicated, in figure VI.2, the angular velocities $\dot{\theta}, \dot{\phi}$ and $\dot{\psi}$ as vectors in what I hope will be agreed are the appropriate directions. FIGURE IV.2 It should be clear that ω_1 is equal to the x_0 -component of $\dot{\theta}$ plus the x_0 -component of $\dot{\phi}\sin\theta$ and that ω_2 is equal to the y_0 -component of $\dot{\theta}$ plus the y_0 -component of $\dot{\phi}\sin\theta$ and that ω_3 is equal to the z_0 -component of $\dot{\theta}$ plus the z_0 -component of $\dot{\psi}$.

For example, the x_0 -component of $\dot{\theta}$ is fairly clearly equal to $\dot{\theta}\cos\theta$, and $\dot{\phi}$ has no x_0 -component. For the x_0 -component of $\dot{\phi}\sin\theta$, you have first of all to project $\dot{\phi}$ on to the x_0y_0 plane, and then find the x_0 -component of that projection, which is $\dot{\phi}\sin\theta\sin\phi$. Thus we obtain $\omega_1 = \dot{\theta}\cos\theta + \dot{\phi}\sin\theta\sin\phi$. 4.2.1 The component ω_2 is a bit more difficult than ω_1 (mostly because the y_2 axis is behind the plane of the paper in figure VI.1), while the component ω_3 is easy. We obtain, after some thought: $\omega_2 = \dot{\theta}\sin\theta\cos\phi - \dot{\phi}\cos\theta$ 4.2.2 and $\omega_3 = \dot{\theta}\sin\theta\sin\phi + \dot{\psi}$ 4.2.3

4.3 Kinetic Energy Most of us are familiar with the formula $\frac{1}{2}I\omega^2$ for the rotational kinetic energy of a rotating solid body. This formula is adequate for simple situations in which a body is rotating about a principal axis, but is not adequate for a body rotating about a nonprincipal axis. I am

going to think of a rotating solid body as a collection of point masses, fixed relative to each other, but all revolving with the same angular velocity about a common axis - and those who believe in atoms assure me that this is indeed the case. (If you believe that a solid is a continuum, you can still divide it in your imagination into lots of small mass elements.) FIGURE IV.3

In figure IV.3, I show just one particle of the rotating body. The position vector of the particle is $\mathbf{r}$. The body is rotating at angular velocity ω . I hope you'll agree that the linear velocity $\mathbf{v}$ of the particle is (now think about this carefully) $\mathbf{v} = \omega \times \mathbf{r}$. The rotational kinetic energy of the solid body is $T = \frac{1}{2} \sum m \mathbf{v} \cdot \mathbf{v} = \frac{1}{2} \sum m (\omega \times \mathbf{r}) \cdot (\omega \times \mathbf{r})$. The triple scalar product is the volume of a parallelepiped, which justifies the next step: $T = \frac{1}{2} \omega \cdot (\sum m \mathbf{r} \times \mathbf{r} \times \omega)$. All particles have the same angular velocity, so: $T = \frac{1}{2} \omega \cdot (\sum m \mathbf{r} \times \mathbf{r} \times \omega)$. Thus we arrive at the following expressions for the rotational kinetic energy: $T = \frac{1}{2} \omega \cdot \mathbf{L}$. 4.3.1 If the body is rotating about a nonprincipal axis, the vectors ω and $\mathbf{L}$ are not parallel (we shall discuss this in more detail in later sections). If it is rotating about a principal axis, they are parallel, and the expression reduces to the familiar $T = \frac{1}{2} I \omega^2$. In matrix notation, this can be written $T = \frac{1}{2} \omega^T \mathbf{I} \omega$. 4.3.2 Here $\mathbf{I}$ is the inertia tensor, ω is a column vector containing the rectangular components of the angular velocity and is its transpose, namely a row vector. That is: $T = \frac{1}{2} \omega^T \mathbf{I} \omega$. 4.3.3 $\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$. 4.3.4 or $T = \frac{1}{2} \omega^T \mathbf{I} \omega$. This expression gives the rotational kinetic energy when the components of the inertia tensor and the angular velocity vector are referred to an arbitrary set of axes. If we refer

them to the principal axes, the off-diagonal elements are zero. I am going to call the principal moments of inertia I_1 , I_2 and I_3 . (I could call them A , B and C , but I shall often use the convention that $A < B < C$, and I don't want to specify at the present which of the three moments is the greatest and which is the greatest, so I'll call them I_1 , I_2 and I_3 , with $I_1 = m(y^2 + z^2)$, etc.). I'll also call the angular velocity components referred to the principal axes ω_1 , ω_2 , ω_3 . Referred, then, to the principal axes, the rotational kinetic energy is $T = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2)$. 4.3.5 I have now dropped the subscript "rot", because in this chapter I am dealing entirely with rotational motion, and so T can safely be understood to mean rotational kinetic energy. We can also now write the kinetic energy in terms of the rates of change of the Eulerian

angles, and the expression we obtain will be useful later when we derive Euler's equations of motion: $T = -\dot{I}_1(\sin\theta\dot{\phi} + \cos\theta\dot{\psi}) + 5I_2(\sin\theta\dot{\phi} - \cos\theta\dot{\psi}) + \dot{I}_3(\cos\theta + 4.3.6$ You will probably want a concrete example in order to understand this properly, so let us imagine that we have a concrete brick of dimensions 10 cm x 15 cm x 20 cm, and of density 4 g cm⁻³, and that it is instantaneously rotating about a body diameter at an angular speed of 6 rad s⁻¹. (Unless the ends of the diameter are fixed in position, the brick will not continue to rotate about the body diameter, which is why I used the word "instantaneously".) 10 cm 20 cm FIGURE IV.4

I hope you'll agree that the mass is 12000 g = 12 kg. The principal moment of inertia about the vertical axis is $I_3 = 12000 \times (10^2 + 7.52 \times 10^2) / 12 = 625,000 \text{ g cm}^2 = 0.0625 \text{ kg m}^2$. Similarly the other principal moments are $I_1 = 0.0500 \text{ kg m}^2$ and $I_2 = 0.0325 \text{ kg m}^2$. The direction cosines of the vector to are $15/20, 20/152 + 202 + 102 \times 152 + 202 + 102 \times 10^2 / 152 + 202 + 102$. Therefore $\dot{\theta} = 3.34252 \text{ rad s}^{-1}$, Hence $T = 0.02103 \text{ J s}^{-2} : 4.45669 \text{ rads}^{-1}, 0)^3 = 2.22834 \text{ rad s}^{-1}$

4.4 Lagrange's Equations of Motion

In Section 4.5 I want to derive Euler's equations of motion, which describe how the angular velocity components of a body change when a torque acts upon it. In deriving Euler's equations, I find it convenient to make use of Lagrange's equations of motion. This will cause no difficulty to anyone who is already familiar with Lagrangian mechanics. Those who are not familiar with Lagrangian mechanics may wish just to understand what it is that Euler's equations are dealing with and may wish to skip over their derivation at this stage. Later in this series, I hope to add a longer chapter on Lagrangian mechanics, when all will be made clear (maybe). In the meantime, for those who are not content just to accept Euler's equations but must also understand their derivation, this section gives a five-minute course in Lagrangian mechanics. To begin with, I have to introduce the idea of generalized coordinates and generalized forces. The geometrical description of a mechanical system at some instant of time can be given by specifying a number of coordinates. For example, if the system consists of just a single particle, you could specify its rectangular coordinates x, y, z , or its cylindrical coordinates ρ, ϕ, z , or its spherical coordinates r, θ, ϕ . Certain theorems to be developed will be equally applicable to any of these, so we can think of generalized coordinates q_1, q_2, q_3 , which could mean any one of the rectangular, cylindrical or spherical set.

In a more complicated system, for example a polyatomic molecule, you might describe the geometry of the molecule at some instant by a set of interatomic distances plus a set of angles between bonds. A fairly large number of distances and angles may be necessary. These distances and angles can be called the generalized coordinates. Notice that generalized coordinates need not always be of dimension L. Some generalized coordinates, for example, may have the dimensions of angle. At this point I raise the question as to whether "angle" is a dimensioned or a dimensionless quantity. In case you have already made up your mind firmly on this point, I present two arguments. The first argument will demonstrate without a shadow of doubt that angle is a dimensionless quantity. The second argument will demonstrate, equally persuasively, that angle is a dimensioned quantity. The first argument is that angle is defined as the ratio of two lengths, namely, it is the ratio of arc length to radius. Thus angle is unquestionably dimensionless. The second argument is that, in stating an angle, it is entirely necessary to state your units. You cannot say merely that the angle is "0.43"; it is entirely necessary to say whether you mean 0.43 degrees or 0.43 radians. Therefore angle unquestionably has dimensions. A most ingenious paradox. For myself, I regard it as most useful to regard angle as a dimensioned quantity, but not as a quantity that can be expressed in terms of M, L and T. Rather, I say, angle has the dimensions of angle. Angle, I say, has dimensions $\angle$. By ascribing to angle its own dimensions, I find that I can considerably extend the usefulness of dimensional analysis. While the generalized coordinates at an instant of time describe the geometry of a system at an instant of time, they alone do not predict the future behaviour of the system. I now introduce the idea of generalized forces. With each of the generalized coordinates there is associated a generalized force. With the generalized coordinate q_i there is associated a corresponding generalized force P_i . It is defined as follows. If, when the generalized coordinate q increases by δq , the work done on the system is $P \delta q$, then P is the generalized force associated with the generalized coordinate q . For example, in our simple example of a single particle, if one of the generalized coordinates is merely the x -coordinate, the generalized force associated with x is the x -component of the force acting on the particle. Note, however, that often one of the generalized coordinates might be an angle. In that case the generalized force associated with it is a torque rather than a force. In other words, a generalized force need not necessarily have the dimensions MLT^{-2} . Before going on to describe Lagrange's equations of

motion, let us remind ourselves how we solve problems in mechanics using Newton's law of motion. We may have a ladder leaning against a smooth wall and smooth floor, or a cylinder rolling down a wedge, the hypotenuse of which is rough (so that the cylinder does not slip) and the smooth base of which is free to obey Newton's third law of motion on a smooth horizontal table, or any of a number of similar problems in mechanics that are visited upon us by our teachers. The way we solve these problems is as follows. We draw a large diagram using a pencil, ruler and compass. Then we mark in red all the forces, and we mark in green all the accelerations. If the problem is a two-dimensional problem, we write $F = ma$ in any two directions; if it is a three-dimensional problem, we write $F = ma$ in any three directions. Usually this is easy and straightforward. Sometimes it doesn't seem to be as easy as it sounds, and we may prefer to solve the problem by Lagrangian methods.

To do this, as before, we draw a large diagram using a pencil, ruler and compass. But this time we mark in blue all the velocities (including angular velocities). Lagrange, in the introduction to his book *La mécanique analytique* (modern French spelling omits the h) pointed out that there were no diagrams at all in his book, since all of mechanics could be done analytically - hence the title of the book. Not all of us, however, are as mathematically gifted as Lagrange, and we cannot bypass the step of drawing a large, neat and clear diagram. Having drawn in the velocities (including angular velocities), we now calculate the kinetic energy, which in advanced texts is often given the symbol T , presumably because potential energy is traditionally written U or V . There would be no harm done if you prefer to write E_k , E_p and E for kinetic, potential and total energy. I shall stick to T , U or V , and E . Now, instead of writing $F = ma$, we write, for each generalized coordinate, the Lagrangian equation (whose proof awaits a later chapter):

4.4.1 The only further intellectual effort on our part is to determine what is the generalized force associated with that coordinate. Apart from that, the procedure goes quite automatically. We shall use it in use in the next section.

That ends our five-minute course on Lagrangian mechanics. 4.5 Euler's Equations of Motion In our first introduction to classical mechanics, we learn that when an external torque acts on a body its angular momentum changes (and if no external torques act on a body its angular momentum does not change.) We learn that the rate of change of angular momentum is equal to the applied torque. In the first simple examples that we typically meet, a

symmetrical body is rotating about an axis of symmetry, and the torque is also applied about this same axis. The angular momentum is just $I\omega$, and so the statement that torque equals rate of change of angular momentum is merely $\tau = I\dot{\omega}$, and that's all there is to it. Later, we learn that $L = I\omega$, where I is a tensor, and L and ω are not parallel. There are three principal moments of inertia, and L , ω and the applied torque τ each have three components, and the statement "torque equals rate of change of angular momentum" somehow becomes much less easy. Euler's equations sort this out, and give us a relation between the components of the and

10 For figure IV.5, I have just reproduced, with some small modifications, figure III. 19 from my notes on this Web site on Celestial Mechanics, where I defined Eulerian angles. Again it is suggested that those who are unfamiliar with Eulerian angles consult Chapter III of Celestial Mechanics. In figure IV.5, $Oxyz$ are space-fixed axes, and $Ox_1y_1z_1$ are the body-fixed principal axes. The axis Oy_1 is behind the plane of your screen; you will have to look inside your monitor to find it. $z \times$ FIGURE IV.5 I suppose an external torque τ_z acts on the body, and I have drawn the components τ_1 and τ_3 . Now let's suppose that the body rotates in such a manner that the Eulerian angle ϕ were to increase by $\delta\phi$. I think it will be readily agreed that the work done on the body is $\tau_3\delta\phi$. This means, following our definition of generalized force in section 4.4, that is the generalized force associated with the generalized coordinate ϕ . Having established that, we can now apply the Lagrangian equation 4.4.1: $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = \tau_3$ 4.5.1 Here the kinetic energy is the expression that we have already established in equation 4.3.6. In spite of the somewhat fearsome aspect of equation 4.3.6, it is quite easy to apply equation 4.5.1 to it. Thus

11 $T = \frac{1}{2}I_1\dot{\theta}^2 \cos^2\theta + \frac{1}{2}I_2\dot{\phi}^2 \sin^2\theta + \frac{1}{2}I_3\dot{\psi}^2$ where I have made use of equation 4.2.3. 4.5.2 Therefore $\tau_3 = I_3\dot{\phi}$. 4.5.3 And, if we make use of equations 4.2.1,2,3, it is easy to obtain Thus equation 4.5.1 becomes: $I_3\dot{\phi} = (I_1 - I_2)\omega \sin^2\theta$. 4.5.4 This is one of the Eulerian equations of motion. Now, although we saw that τ_3 is the generalized force associated with the coordinate ϕ , it will be equally clear that τ_z is not the generalized force associated with θ , nor is τ_2 the generalized force associated with ψ . However, we do not have to think about what the generalized forces associated with these two coordinates are; it is much easier than that. To obtain the remaining two Eulerian equations, all that is necessary is to carry out a cyclic permutation of the subscripts in equation

4.5.5. Thus the three Eulerian equations are: $I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = \tau_1$, $I_2 \dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = \tau_2$, $I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = \tau_3$. 4.5.6 $I_1 \dot{\omega}_1 = \tau_1$, $I_2 \dot{\omega}_2 = \tau_2$, $I_3 \dot{\omega}_3 = \tau_3$. 4.5.7 $I_3 \dot{\omega}_3 = \tau_3$. 4.5.8 These take the place of $\tau = I \ddot{\theta}$, which we are more familiar with in elementary problems in which a body is rotating about a principal axis and a torque is applied around that principal axis. If there are no external torques acting on the body, then we have Euler's equations of free rotation of a rigid body: $I_1 \dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3$, $I_2 \dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1$, $I_3 \dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2$. 4.5.9

12 Example. $I_1 \dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3$, $I_2 \dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1$, $I_3 \dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2$. 4.5.10 $I_3 \dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2$. 4.5.11 ω_3 FIGURE IV.6 Figure IV.6 shows a disc of mass m , radius a , spinning at a constant angular speed ω_0 about an axle that is inclined at an angle θ to the normal to the disc. I have drawn three body-fixed principal axes. The x - and y -axes are in the plane of the disc; the direction of the x -axis is chosen so that the axle (and hence the vector ω_0) is in the zx -plane. The disc is evidently unbalanced and there must be a torque on it to maintain the motion. Since ω_0 is constant, all components of τ are zero, so that Euler's equations are $\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3/I_1$, $\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1/I_2$, $\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2/I_3$. $\omega_1 = \omega_0 \sin \theta \cos \phi$, $\omega_2 = \omega_0 \sin \theta \sin \phi$, $\omega_3 = \omega_0 \cos \theta$. $I_1 = \frac{1}{2}ma^2$, $I_2 = \frac{1}{2}ma^2$, $I_3 = \frac{1}{2}ma^2$. Now $\dot{\omega}_1 = \omega_0 \sin \theta \cos \phi \dot{\phi}$, $\dot{\omega}_2 = \omega_0 \sin \theta \sin \phi \dot{\phi}$, $\dot{\omega}_3 = 0$. Therefore $\dot{\phi} = 0$, and $\phi = 0$. $\tau_x = -I_2 \dot{\omega}_2 \omega_3 = -\frac{1}{2}ma^2 \omega_0^2 \sin^2 \theta \cos \theta$, $\tau_y = -I_3 \dot{\omega}_3 \omega_1 = 0$, $\tau_z = -I_1 \dot{\omega}_1 \omega_2 = -\frac{1}{2}ma^2 \omega_0^2 \sin^2 \theta \sin \theta \cos \theta$. (Check, as always, that this expression is dimensionally correct.) on the disc is in the negative y -direction. Thus the torque acting

13 Can you reconcile the fact that there is a torque acting on the disc with the fact that it is moving with constant angular velocity? Yes, most decidedly! What is not constant is the angular momentum L , which is moving around the axle in a cone such that $L = -z^2 j$, where j is the unit vector along the y -axis. 4.6 Free Rotation of a Rigid Asymmetric Top By "asymmetric top" I mean a body whose three principal moments of inertia are unequal. While we often think of a "top" as a symmetric body spinning on a table, in this section the "top" will not necessarily be symmetric, and it will not be in contact with any table, nor indeed subjected to any external forces or torques. A complete description of the motion of an asymmetric top is quite complicated, and therefore all that we shall attempt in this chapter is a qualitative description of certain aspects of the motion. That our description is going to be "qualitative" does not by any means imply that this section is not going to be replete with equations or that we can give our poor brains a rest. The first point that we can make is that, provided that no external torques act on the body, its angular momentum L is constant in magnitude and direction. A second point is that, provided the body is rigid and has no internal degrees of freedom, the

rotational kinetic energy T is constant. I deal briefly with nonrigid bodies in section 4.7. Although the angular velocity vector ω is by no means fixed in either magnitude and direction, and the body can tumble over and over, these two conditions impose some constraints of the magnitude and direction of ω . We are going to examine these two conditions to see what constraints are imposed on ω . One of the things we shall find is that rotation of a body about a principal axis of greatest or of least moment of inertia is stable against small displacements, whereas rotation about the principal axis of intermediate moment of inertia is unstable. Absence of an external torque means that the angular momentum is constant: $L^2 = L_1^2 + L_2^2 + L_3^2 = \text{constant}$, so that, at all times, $L_1^2 + L_2^2 + L_3^2 = L^2$. Thus, for a given L , the angular velocity components always satisfy $L_1^2 + L_2^2 + L_3^2 = L^2$. 4.6.1 4.6.2 4.6.3

14 That is to say, the angular velocity vector is constrained such that the tip of the vector is always on the surface of an ellipsoid of semi axes L/I_1 , L/I_2 , L/I_3 . In addition to the constancy of angular momentum, the kinetic energy is also constant: $\frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2 = T$. 4.6.4 Thus the tip of the angular velocity vector must also be on the surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. 4.6.5 This ellipsoid (which is similar in shape to the momental ellipsoid) has semi axes $a = L/I_1$, $b = L/I_2$, $c = L/I_3$. Thus, however the body tumbles over and over, it is constrained in magnitude and direction so that its tip is on the curve where these two ellipsoids intersect. Suppose, for example, that we have a rigid body with $I_1 = 0.2 \text{ kgm}^2$, $I_2 = 0.3 \text{ kgm}^2$, $I_3 = 0.5 \text{ kgm}^2$ and that we set it in motion such that the angular momentum and kinetic energy are $L = 4 \text{ Js}$, $T = 20 \text{ J}$. (The angular momentum and kinetic energy will be determined by the magnitude and direction of the initial velocity vector by which it is set in motion.) The tip of ω is constrained to be on the curve of intersection of the two ellipsoids $x^2 + y^2 + z^2 = 16$ and $\frac{x^2}{16} + \frac{y^2}{24} + \frac{z^2}{40} = 1$. 4.6.6 202 13.32 82 $x^2 + y^2 + z^2 = 16$ and $\frac{x^2}{16} + \frac{y^2}{24} + \frac{z^2}{40} = 1$. 4.6.7 14.142 11.552 8.942 It is not easy (or I don't find it so) to imagine what this curve of intersection looks like in three-dimensional space. It is not a plane curve. If you are better with computers than I am you might be able to draw a three-dimensional graph. In MATLAB, for example, I believe the function `plot3` will do it. If any reader manages to do it, let me know (universeuvm.uvic.ca), and we'll see if we can include your graph, with

15 acknowledgment, of course, in these notes. In case it's of any help, you

the rotation is stable. Case II. Rotation about the axis of greatest moment of inertia. That is, we'll take $I_3 = 5$, $I_2 = 2$, $I_1 = 3$. Since I_3 is the greatest moment of inertia, each of the ratios I_3/I_1 and I_3/I_2 are greater than 1, and $I_3/I_1 > I_3/I_2$ and $I_3/I_2 > 1$. The two ellipsoids are

$2.50f$: and $1.58C_1$ I'll try and sketch these: $2.0) 2 + (1.67f/)^2 2.0) 2 + (1.29f/)^2 2.2 4.6.14 4.6.15 // I I I I I I L \ \ \ J.29 \ \ \ 1.58 2.50$

FIGURE IV.9 Again, and for the same reason as for Case I, we see that this motion is stable.

Case III. Rotation about the intermediate axis. That is, we'll take $I_3 = 3$, $I_2 = 5$, $I_1 = 2$. This time I_3/I_1 is less than 1 and I_3/I_2 is less than 1, and $I_3/I_1 > I_3/I_2$ and $I_3/I_2 < 1$. The two ellipsoids are $2.2 2.0 (0.60F)$: $(1.50n)$: F : $4.6.16$ and $2.2 2.0 (0.60F)$: $(1.50n)$: F : $4.6.16$ and $2.2 2.0 (0.60F)$: $(1.50n)$: F : $4.6.16$ and $2.2 2.0 (0.60F)$: $(1.50n)$: F : $4.6.16$

FIGURE IV. 10 Unlike the situation for Cases I and II, in which the two ellipsoids touch at only a single point, the two ellipses for Case III intersect in the curve shown as a dotted line in figure

IV. 10. Thus it is not restricted to lying along the z-axis, but it can move anywhere along the dotted line. The motion, therefore, is not stable. You should experiment by throwing a body in the air in such a manner as to let it spin around one of its principal axes. A rectangular block will do, though the effect is particularly noticeable with something like a table-tennis bat or a tennis racket. Here is another approach to reach the same result. We imagine an asymmetric top spinning about one of its principal axes with angular velocity $\omega = \omega_0$. It is then given a small perturbation, so that its angular velocity is now $\omega = \omega_0 + \delta\omega$. Here the "hatted" quantities are the unit orthogonal vectors; and r_1 are supposed small compared with ω_0 . Euler's equations are $\hat{L}_i = r_1(I_i - I_j)\hat{\omega}_j$, $4.6.19$ $12 q: Oz413 - 1$), $4.6.20$ $1, \&z: 4.6.21$ If $gr_1 \ll \&$, then ω is approximately constant. Elimination of r_1 from the first two equations yields $\ddot{\theta} + \frac{I_2 - I_3}{I_1(I_2 - I_3)}\omega_0^2 \theta = 0$ $4.6.22$ Elimination of θ instead results in a similar equation in r_1 . If I_3 is either the largest or the smallest of the three moments of inertia, the two parentheses in the denominator have the same sign, so the expression in the brackets is positive. Equation 4.6.22 is then the equation for simple harmonic motion, and the motion is stable. If, however, I_3 is intermediate between the other two, the two parentheses have

opposite sign, and the expression in brackets is negative. In that case r increases exponentially, and the motion is unstable.

21 4.7 Nonrigid rotator 1 2 The rotational kinetic energy of a body rotating about a principal axis is $\frac{1}{2} I \omega^2$, where I is the moment of inertia about that principal axis, and the angular momentum is $L = I\omega$. (For rotation about a nonprincipal axis, see section 4.3.) Thus the rotational kinetic energy can be written as $L^2/(2I)$. When an asymmetric top is rotating about a nonprincipal axis, the body experiences internal stresses, which, if the body is nonrigid, result in periodic strains which periodically distort the shape of the body. As a result of this, rotational kinetic energy becomes degraded into heat; the rotational kinetic energy of the body gradually decreases. In the absence of external torques, however, the angular momentum is constant. The expression $L^2/(2I)$ for the kinetic energy shows that the kinetic energy is least for a given angular momentum when the moment of inertia is greatest. Thus eventually the body rotates about its principal axis of greatest moment of inertia. After that, it no longer loses kinetic energy to heat, because, when the body is rotating about a principal axis, it is no longer subject to internal stresses. The time taken (the "relaxation time") for a body to reach its final state of rotation about its principal axis of greatest moment of inertia depends, among other things, on how fast the body is rotating. A fast rotator will reach its final state relatively soon, whereas it takes a long time for a slow rotator to reach its final state. Thus it is not surprising to find that, among the asteroids, most of the fast rotators are principal axis rotators, whereas many slow rotators are also nonprincipal axis rotators. There are, however, a few fast rotators that are still rotating about a nonprincipal axis. It is assumed that such asteroids may have suffered a collision in the recent past.

4.8 Force-free Motion of a Symmetric Top Notation: I_1, I_2, I_3 are the principal moments of inertia. I_3 is the unique moment. If it is the largest of the three, the body is an oblate symmetric top; if it is the smallest, it is a prolate spherical top. Ox, Oy, Oz are the corresponding body-fixed principal axes. $\omega_1, \omega_2, \omega_3$ are the components of the angular velocity vector ω with respect to the principal axes. In the analysis that follows, we are going to have to think about three vectors. There will be the angular momentum vector L , which, in the absence of external torques, is fixed in magnitude and in direction in laboratory space. There will be the direction of the axis of symmetry, the Oz axis, which is fixed in the body, but not

necessarily in space, unless the body happens to be rotating about its axis of symmetry; we'll denote a unit vector in this direction by $\hat{z}_0$. And there will be the instantaneous angular velocity vector ω which is neither space- nor body-fixed.

22 What we are going to find is the following. We shall find that ω precesses in the body about the body-fixed symmetry axis in a cone called the body cone. The angle between ω and $\hat{z}_0$ is constant (we'll be calling this angle θ_0 , and the magnitude $|\omega|$ is constant. We shall find that the sense of the precession is the same as the sense of the spin if the body is oblate, but opposite if it is prolate. The direction of the symmetry axis, however, is not fixed in space, but it precesses about the space-fixed angular momentum vector L in another cone. This cone is narrower than the body cone if the body is oblate, but broader than the body cone if the body is prolate. The net result of these two precessional motions is that ω precesses in space about the space-fixed angular momentum vector in a cone called the space cone. For an oblate top, the semi vertical angle of the space cone can be anything from 0° to 90° ; for an oblate top, however, the semi vertical angle of the space cone cannot exceed $19^\circ 28'$. That's quite a lot to take in in one breath! We can start with Euler's equations of motion for force-free rotation of a symmetric top: 4.8.1 $I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_2 \omega_3 = 0$, 4.8.2 $I_2 \dot{\omega}_2 + (I_3 - I_1)\omega_1 \omega_3 = 0$. 4.8.3 From the first of these we obtain the result $\dot{\omega}_3 = 0$. 4.8.4 For brevity, I am going to let $\omega_3 = \Omega$ although in a moment Ω will have a physical meaning. Equations 4.8.1 and 4.8.2 become: and Eliminate $\dot{\omega}_2$ from these to obtain 4.8.5 $I_1 \dot{\omega}_1 + (I_3 - I_2)\omega_1 \Omega = 0$ 4.8.6 $I_2 \dot{\omega}_1 - (I_3 - I_1)\omega_1 \Omega = 0$ 4.8.7 4.8.8

23 This is the equation for simple harmonic motion and its solution is $\omega_1 = \theta_0 \cos(f/t + \delta)$, 4.8.9 in which θ_0 and δ , the two constants of integration, whose values depend on the initial conditions in the usual fashion, are the amplitude and initial phase angle. On combining this with equation 4.8.6, we obtain $\dot{\omega}_2 = \theta_0 \sin(f/t + \delta)$. 4.8.10 From these we see that $(\dot{\omega}_1^2 + \dot{\omega}_2^2)^{1/2}$, which is the magnitude of the component of ω in the xy-plane, is constant, equal to $\theta_0 \Omega$; and since ω_3 is also constant, it follows that $(\dot{\omega}_1^2 + \dot{\omega}_2^2 + \omega_3^2)^{1/2}$, which is the magnitude of ω , is also constant. The cosine of the angle θ between ω and $\hat{z}_0$ is $\omega_3 / (\dot{\omega}_1^2 + \dot{\omega}_2^2 + \omega_3^2)^{1/2}$, and its sine is $\theta_0 \Omega / (\dot{\omega}_1^2 + \dot{\omega}_2^2 + \omega_3^2)^{1/2}$ so that θ is constant. Equations 4.8.9 and 4.8.10 tell us, then, that the vector ω is precessing around the symmetry axis at an angular speed f . Making use of equation 4.8.5, we find that $(I_1 \dot{\omega}_1^2 + I_2 \dot{\omega}_2^2 + I_3 \omega_3^2)^{1/2} = \text{constant}$

4.8.11 0) ($I_3 - I_1$)0) If we take the direction of the z_0 axis to be the direction of the component of ω along the symmetry axis, then $\mathbf{F}_I$ is in the same direction as $\mathbf{Z}_0$ if $I_3 > I_1$ (that is, if the top is oblate) and it is in the opposite direction if the top is prolate. The situation for oblate and prolate tops is shown in figure IV. 11. Oblate Prolate FIGURE IV. 11

24 We have just dealt with how the instantaneous axis of rotation precesses about the body-fixed symmetry axis, describing the body cone of semi vertical angle θ . Now we are going to consider the precession of the body-fixed symmetry axis about the space-fixed angular momentum vector $\mathbf{L}$. I am going to make use of the idea of Eulerian angles for expressing the orientation or one three-dimensional set of axes with respect to another. If you are not already familiar with Eulerian angles or would like a refresher, you can go to <http://astrowww.phys.uvic.ca/~tatum/celmechs/celm3.pdf> especially section 3.7, page 29. Recall that we are using $Ox_0 y_0 z_0$ for body-fixed coordinates, referred to the principal axes. I shall use $Oxyz$ for space-fixed coordinates, and there is no loss of generality if I choose the Oz axis to coincide with the angular momentum vector $\mathbf{L}$. Let me try to draw the situation in figure IV. 12. The axes $Oxyz$ are the space-fixed axes. I have drawn only the x - and z -axes, and I leave the y -axis to your imagination. The axes $Ox_0 y_0 z_0$ are the body-fixed principal axes. The angular momentum vector $\mathbf{L}$ is directed along the axis Oz . The symmetry axis of the body is directed along the axis Oz_0 . The Eulerian angles of the body-fixed axes relative to the space fixed axes are (ϕ, θ, ψ) , formed in the usual way by: First, a rotation by ϕ about Oz to define an intermediate set of axis Ox' . Second, a rotation by θ about Ox' to form Oz_0 , which is the symmetry axis. Third, a rotation by ψ about Oz_0 to form the body-fixed principal axes. (I have not drawn the axis Oy_0 , which lies behind the plane of your screen.) $Z_0: \mathbf{L}$ FIGURE IV.12

25 Figure IV. 12 merely shows the instantaneous orientation of the body-fixed axes relative to the space-fixed axes at some time. We are not supposing that the body is rotating about its symmetry axis. Indeed we have already found that, if the body is not rotating about its symmetry axis, the angular velocity vector ω precesses about the symmetry axis at a rate C^2 , the magnitude ω of ω and the angle θ between ω and the symmetry axis remaining constant. In figure IV.13 I add the vector ω and I indicate its precessional motion around the symmetry axis at a rate C^2 . The vectors $\mathbf{L}$

and ω are each at right angles to the Ox' axes - i.e. they are in the $Oy'z'$ plane. If you look towards O down the Oz' axis (figure IV.14) you can see that the components of L and ω with respect to the primed axes are: $L_x = 0$, $L_y = L \sin \theta$, $L_z = L \cos \theta$ 4.8.12 and $\omega_x = 0$, $\omega_y = \omega \sin \theta$, $\omega_z = \omega \cos \theta$. 4.8.13 x FIGURE IV. 13

26 Z : Z 0 FIGURE IV. 14 From these, together with $L_y = I \omega_y$, and $L_z = I \omega_z$, we obtain $I \tan \theta = I \omega \tan \theta$. For an oblate symmetric top, $I_3 > I$, $\therefore \theta > 0$. For a prolate symmetric top, $I_3 < I$, $\therefore \theta < 0$. 4.8.14 Now when a body rotates through an angle ω in time t , the three Eulerian angles also change, and the angle ω can be regarded as the vector sum of the three infinitesimal changes in the Eulerian angles: $\therefore d\omega = d\phi + d\theta + d\psi$, or $\omega = d\phi + d\theta + d\psi$. 4.8.15 (Infinitesimal angular displacements do not conform to all the properties of vectors, but they conform sufficiently for our present purposes. Technically they are pseudovectors.) The components of ω and $\dot{\phi}$ along Oy' are each zero, and therefore the component of ω along Oy' is equal to the component of $\dot{\phi}$ along Oy' . $\therefore \omega \cos \theta = \dot{\phi} \sin \theta$. 4.8.16 In summary, then:

27 1. The instantaneous axis of rotation, which makes an angle θ with the symmetry axis, precesses around it at angular speed $I_3 - I$ - $\omega \cos \theta$, 4.8.17 which is in the same sense as ω if the top is oblate and opposite if it is prolate. 2. The symmetry axis makes an angle θ with the space-fixed angular momentum vector L , where $\tan \theta = I \omega \sin \theta$. 4.8.18 For an oblate top, $\theta < \theta_0$. For a prolate top, $\theta > \theta_0$. 3. The speed of precession of the symmetry axis about L is $(I_3 - I) \sin \theta$, 4.8.19 $\sin \theta$ or, by elimination of θ between 4.8.18 and 4.8.19, $I_3/2I$ ($1 + I_3/I \cos 2\theta$). 4.8.20 The net result of this is that ω precesses about L at a rate $(I_3 - I) \sin \theta$ in the space cone, which has a semi-vertical angle θ - θ_0 for an oblate rotator, and $\theta_0 - \theta$ for a prolate rotator. The space cone is fixed in space, while the body cone rolls around it, always in contact, ω being a mutual generator of both cones. If the rotator is oblate, the space cone is smaller than the body cone and is inside it. If the rotator is prolate, the body cone is outside the space cone and can be larger or smaller than it. Write $c = I_3/I$ 4.8.21 for the ratio of the principal moments of inertia. Note that for a pencil, $c = 0$; for a sphere, $c = 1$; for a plane disc or any regular plane lamina, $c = 2$. (The last of these follows from the perpendicular axes theorem - see Chapter 2.) The range of c , then, is from 0 to 2, 0 to 1 being prolate, 1 to 2 being oblate. Equations 4.8.17 and 4.8.20 can be written

28 --: $(c-1) \cos \theta$ 4.8.22 and $(= [1 + (c^2 - 1) \cos^2 \theta]^{1/2})$. 4.8.23 Figures IV. 15 and IV. 16 show, for an oblate and a prolate rotator respectively, the instantaneous rotation vector ω precessing around the body-fixed symmetry axis at a rate C_2 in the body cone of semi vertical angle θ ; the symmetry axis precessing about the space-fixed angular momentum vector L at a rate $\dot{\phi}$ in a cone of semi vertical angle θ_0 (which is less than θ for an oblate rotator, and greater than θ for a prolate rotator; and consequently the instantaneous rotation vector ω precessing around the space-fixed angular momentum vector L at a rate $\dot{\phi}$ in the space cone of semi vertical angle $\theta_0 - \theta$ (oblate rotator) or $\theta + \theta$ (prolate rotator). Angular momentum L Symmetry axis Space cone Body

FIGURE IV. 15
29 Symmetry axis: Angular momentum L Body FIGURE IV. 16 One can see from figures IV. 15 and 16 that the angle between L and ω is limited for an oblate rotator, but it can be as large as 90° for a prolate rotator. The angle between L and ω is $\theta_0 - \theta$ (which is negative for an oblate rotator). We have $\tan \theta_0 - \tan \theta = (1 - c) \tan \theta = 4.8.24 \tan(\theta_0 - \theta) = 1 + \tan \theta \tan \theta c + \tan^2 \theta$ By calculus this reaches a maximum value of for $\tan \theta = J^-$. For a rod or pencil (prolate), in which $c = 0$, the angle between L and ω can be as large as 90° . Recalling exactly what are meant by the vectors L and ω , the reader should try now

30 and imagine in his or her mind's eye a pencil rotating so that L and ω are at right angles. The spin vector ω is along the length of the pencil and the angular momentum vector L is at right angles to the length of the pencil. For an oblate rotator, the angle between L and ω is limited. The most oblate rotator is a flat disc or any regular flat lamina. The parallel axis theorem shows that for such a body, $c = 2$. The greatest angle between L and ω for a disc occurs when $\tan \theta = x/2$ (or $\theta = 54^\circ 44'$), and then $\tan(\theta_0 - \theta) = 1/x/8$, $\theta_0 - \theta = 19^\circ 28'$. In figures IV. 17, 18 and 19, I illustrate some of these results graphically. Figure IV. 17 shows $\dot{\phi}/\omega$ (the rate of precession of the spin vector ω around the symmetry axis) in units of ω , as a function of I_3/I , for several values of θ , the angle between ω and the symmetry axis. This is calculated from equation 4.8.17. The ratio I_3/I goes from 0 for a pencil through 1 for a sphere to 2 for a disc. PENCIL SPHERE 1 0.8 0.6 0.4 0.2 0 -0.2 -0.4 -0.6 -0.8 -1 0 PROLATE OBLATE

FIGURE IV. 17 0.5 1 1.5 2 I_3/I 1 DISC ω ω 30 60 90 Figure IV. 18 shows $\dot{\phi}$ (the rate of precession of the symmetry axis around the angular

momentum vector L) in units of $c\omega$, as a function of I_3/I , also for several values of α , the angle between ω and the symmetry axis. Recall that the angle θ between the symmetry axis and the angular momentum vector is given by equation 4.8.19.) Figure IV.18 is calculated from equation 4.8.20.

31 PENCIL 2 1.8 1.6 1.4 1.2 1 0.8 0.6 0.4 0.2 0 0 SPHERE PFOLATE
OCLATE FIGURE IV.18 0.5 1 1.5 2 I_3/I 1 DISC 0 0 30 60 90 Figure IV. 19
shows the greatest possible value of the angle, in degrees, between L and ω
as a function of I_3/I . PENCIL SPHERE DISC 90 PROLATE OBLATE S 80
70 60 50 40 30 20 10 0 -10 -20 0 0.2 0.4 0.6 0.8 1 I_3/I 1 FIGURE IV.19 1.2
1.4 1.6 1.8 2

32 Our planet Earth is approximately an oblate spheroid, its dynamical ellipticity $(I_3 - I)/I$ being about 3.285×10^{-3} . It is not rotating exactly about its symmetry axis; the angle α between ω and the symmetry axis being about one fifth of an arcsecond, which is about six metres on the surface. The rotation period is one sidereal day (which is a few minutes shorter than 24 solar hours.) Equation 4.8.17 tells us that the spin axis precesses about the symmetry axis in a period of about 304 days, all within the area of a tennis court. The actual motion is a little more complicated than this. The period is closer to 432 days because of the nonrigidity of Earth, and superimposed on this is an annual component caused by the annual movement of air masses. This precessional motion of a symmetric body spinning freely about an axis inclined to the symmetry axis gives rise to variations of latitude of amplitude about a fifth of an arcsecond. It is not to be confused with the 26,000 year period of the precession of the equinoxes, which is caused by external torques from the Moon and the Sun.

4.9 Centrifugal and Coriolis Forces We are usually told in elementary books that there is "no such thing" as centrifugal force. When a satellite orbits around Earth, it is not held in equilibrium between two equal and opposite forces, namely gravity acting towards Earth and centrifugal force acting outwards. In reality, we are told, the satellite is accelerating (the centripetal acceleration); there is only one force, namely the gravitational force, which is equal to the mass times the centripetal acceleration. Yet when we drive round a corner too fast and we feel ourselves flung away from the centre of curvature of our path, the "centrifugal force" certainly feels real enough, and indeed we can solve problems referred to rotating coordinate systems as if there "really" were such a thing as

"centrifugal force". Let's look at an even simpler example, not even involving rotation. A car is accelerating at a rate a towards the right. See figure IV.20 - but forgive my limited artistic abilities. Drawing a motor car is somewhat beyond my skills. FIGURE IV.20

33 There is a plumb-bob hanging from the roof of the car, but, because of the acceleration of the car, it is not hanging vertically. Some would say that there are but two forces on the plumb-bob - its weight and the tension in the string - and, as a result of these, the bob is accelerating towards the right. By application of $F = ma$ it is easily possible to find the tension in the string and the angle that the string makes with the vertical. The passenger in the car, however, sees things rather differently (figure IV.21 .) ma / FIGURE IV.21 To the passenger in the car, nothing is accelerating. The plumb-bob is merely in static equilibrium under the action of three forces, one of which is a force ma towards the left. To the question "But what is the agent that is causing this so-called force?" I counter with the question "What is the agent that is causing the downward force that you attribute to some mysterious 'gravity' "? It seems that, when referred to the reference frame of figure IV.20, there are only two forces, but when referred to the accelerating reference frame of figure IV.21, the system can be described perfectly well by postulating the existence of a force ma pulling towards the left. This is in fact a principle in classical mechanics, known as d'Alembert's principle, whereby, if one refers the description of a system to an accelerating reference frame, one can replace an acceleration with a force in the opposite direction. It results in a perfectly valid description of the behavior of a system, and will accurately predict how the system will behave. So who's to say which forces are "real" and which are "fictitious", and which reference frame is better than another?

34 The situation is similar with respect to centrifugal force. If you consider a satellite in orbit around Earth, some would say that there is only one force acting on the satellite, namely the gravitational force towards Earth. The satellite, being in a circular orbit, is accelerating towards the centre of the circle, and all is as expected - $F = ma$. The acceleration is the centripetal acceleration (peto - I desire). An astronaut on board the satellite may have a different point of view. He is at a constant distance from Earth, not accelerating; he is in static equilibrium, and he feels no net force on him whatever - he feels weightless. If he has been taught that Earth exerts a

gravitational force, then this must be balanced by a force away from Earth. This force, which becomes apparent when referred to a corotating reference frame, is the centrifugal force (lingo - I flee, like a fugitive). It would need a good lawyer to argue that the invisible gravitational force towards Earth is a real force, while the equally imaginary force acting away from Earth is imaginary. In truth, all forces are "imaginary" - in that they are only devices or concepts used in physics to describe and predict the behaviour of matter. I mentioned earlier a possible awful examination question: Explain why Earth bulges at the equator, without using the term "centrifugal force". Just thank yourself lucky if you are not asked such a question! People who have tried to answer it have come up with some interesting ideas. I have heard (I don't know whether it is true) that someone once offered a prize of \$1000 to anyone who could prove that Earth is rotating, and that the prize has never been claimed! Some have tried to imagine how you would determine whether Earth is rotating if it were the only body in the universe. There would be no external reference points against which one could measure the orientation of Earth. It has been concluded (by some) that even to think of Earth rotating in the absence of any external reference points is meaningless, so that one certainly could not determine how fast, or even whether and about what axis, Earth was rotating. Since "rotation" would then be meaningless, there would be no centrifugal force, Earth would not bulge, nor would the Foucault pendulum rotate, nor would naval shells deviate from their paths, nor would cyclones and anticyclones exist in the atmosphere.. Centrifugal force comes into existence only when there is an external universe. It is the external universe, then, revolving around the stationary Earth, that causes centrifugal force and all the other effects that we have mentioned. These are deep waters indeed, and I do not pursue this aspect further here. We shall merely take the pragmatic view that problems in mechanics can often be solved by referring motions to a corotating reference frame, and that the behaviour of mechanical systems can successfully and accurately be described and predicted by postulating the "existence" of "inertial" forces such as centrifugal and Coriolis forces, which make themselves apparent only when referred to a rotating frame. Thus, rather than involving ourselves in difficult questions about whether such forces are real, we shall take things easy with just a few simple equations.

35 P FIGURE IV.22 5; = Oxyz is an inertial reference frame (i.e. not

"hat" ($\hat{}$) denotes a unit vector. We have $\mathbf{c} \times \mathbf{r} = \mathbf{r} \sin \theta \hat{\mathbf{z}}$ and hence $\mathbf{c} \times (\mathbf{c} \times \mathbf{r}) = \mathbf{r} \sin^2 \theta - \mathbf{r} \cos^2 \theta = \mathbf{r} (\sin^2 \theta - \cos^2 \theta)$ is a unit vector directed towards the z-axis. Notice that the point P is moving at angular speed ω in a small circle of radius $r \sin \theta$. The expression $\mathbf{a} = \mathbf{c} \times (\mathbf{c} \times \mathbf{r}) = \omega^2 r (\sin^2 \theta - \cos^2 \theta) \hat{\mathbf{z}}$, then, is just the familiar centripetal acceleration, of magnitude $\omega^2 r \sin^2 \theta$, directed towards the axis of rotation. We could also think of $\mathbf{c} \times (\mathbf{c} \times \mathbf{r})$ as a triple vector product. We recall that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$ so that $\mathbf{r} \times (\mathbf{r} \times \mathbf{c}) = (\mathbf{r} \cdot \mathbf{c})\mathbf{r} - r^2 \mathbf{c}$. That is $\mathbf{r} \times (\mathbf{r} \times \mathbf{c}) = r \cos \theta \mathbf{r} - r^2 \mathbf{c}$. This can be illustrated by the vector diagram shown in figure IV.23. The vectors are drawn in green, in accordance with my convention of red, blue and green for force, velocity and acceleration respectively.

FIGURE IV.23

38 However, equation 4.9.4 also tells us that, if a particle is moving with velocity $\mathbf{v}'$ with respect to Z' , it has an additional acceleration with respect to Z of $2\boldsymbol{\omega} \times \mathbf{v}'$, which is at right angles to $\mathbf{v}'$ and to $\boldsymbol{\omega}$. This is the Coriolis acceleration. The converse of equation 4.9.4 is $\mathbf{a}' = \mathbf{a} + \mathbf{c} \times (\mathbf{r} \times \mathbf{c}) + 2\mathbf{v}' \times \boldsymbol{\omega}$.

4.9.6 If a particle has a force $\mathbf{F} = m\mathbf{a}$ acting on it with respect to Z , when referred to Z' it will appear that the particle is moving under the influence of three forces: the "true force" $\mathbf{F}$, the centrifugal force acting away from the axis of rotation, and the Coriolis force, which acts only when the particle is in motion with respect to Z' , and which is at right angles to both $\mathbf{v}'$ and to $\boldsymbol{\omega}$: $\mathbf{F}' = \mathbf{F} + m\mathbf{c} \times (\mathbf{r} \times \mathbf{c}) + 2m\mathbf{v}' \times \boldsymbol{\omega}$.

4.9.7 It is worth now spending a few moments thinking about the direction of the Coriolis force $2m\mathbf{v}' \times \boldsymbol{\omega}$. Earth is spinning on its axis with a period of 24 sidereal hours (23 h and 56 m of solar time.). The vector $\boldsymbol{\omega}$ is directed upwards through the north pole. Now go to somewhere on Earth at latitude 45° N. Fire a naval shell to the north. To the east. To the south. To the west. Now go to the equator and repeat the experiment. Go to the north pole. There you can fire only due south. Repeat the experiment at 45° south, and at the south pole. Each time, think about the direction of the vector $2m\mathbf{v}' \times \boldsymbol{\omega}$. If your thoughts are to my thoughts, your mind to my mind, you should conclude that the shell veers to the right in the northern hemisphere and to the left in the southern hemisphere, and that the Coriolis force is zero at the equator. As air rushes out of a high pressure system in the northern hemisphere, it will swirl clockwise around the pressure centre. As it rushes in to a low pressure system, it will swirl counterclockwise. The opposite situation will happen in the southern

hemisphere. You can think of the Coriolis force on a naval shell as being a consequence of conservation of angular momentum. Go to 45°N and point your naval gun to the north. Your shell, while waiting in the breech, is moving around Earth's axis at a linear speed of $\omega R \sin 45^\circ$, where R is the radius of Earth. Now fire the shell to the north. By the time it reaches latitude 50°N , it is being carried around Earth's axis in a small circle of radius only $R \sin 40^\circ$. In order for angular momentum to be conserved, its angular speed around the axis must speed up - it will be deviated towards the east. Now try another thought experiment ((Bebanen rufun.)). Go to equator and build a tall tower. Drop a stone from the top of the tower. Think now about the direction of the vector $2\mathbf{mv} \times \boldsymbol{\omega}$. I really mean it - think hard. Or again, think about conservation of angular momentum. The stone drops closer to Earth's axis of rotation. It must conserve angular momentum. It falls to the east of the tower (nay, not to the west!).

39 x North $\rightarrow$ East FIGURE IV.24 The two forces on the stone are its weight mg and the Coriolis force. Earth's spin vector $\boldsymbol{\omega}$ is to the north. The Coriolis force is at right angles to the stone's velocity. If we resolve the stone's velocity into a vertically down component v_y and a horizontal east component v_x , the corresponding components of the Coriolis force will be $m\omega v_y$ to the east and $m\omega v_x$ upward. However, I'm going to assume that $\omega \ll \omega_0$ and the only significant Coriolis force is the eastward component $m\omega v_y$, which I have drawn. Another way of stating the approximation is to say that the upward component of the Coriolis force is negligible compared with the weight mg of the stone. After dropping for a time t , the y-coordinate of the stone is found in the usual way from $y = \frac{1}{2}gt^2$, $y = \frac{1}{2}gt^2$, and the x-coordinate is found from $\ddot{x} = \omega v_y$: $\ddot{x} = \omega \frac{dy}{dt}$, $x = \frac{1}{2}\omega y$. Thus you can find out how far to the east it has fallen after two seconds, or how far to the east it has fallen if the height of the tower is 100 metres. The equation to the trajectory would be the t -eliminant, which is $x^2 = \frac{1}{2}\omega y^2$.

40 For Earth, $\omega = 7.292 \times 10^{-5} \text{ rad/s}$, and at the equator $\omega R = 9.780 \text{ m/s}$, so that $\frac{1}{2}\omega R = 1.197 \times 10^{-4} \text{ m/s}^2$. The path is graphed in figure IV.25 for a 100-metre tower. The horizontal scale is exaggerated by a factor of about 104. FIGURE IV.25 100 90 80, 70 E ωR , 8 so 40 - 30 20 10 0 0 2 4 6 8 10 12

Displacement to east, millimetres I once asked myself the question whether a migrating bird could navigate by using the Coriolis force. After all, if it were

flying north in the northern hemisphere, it would experience a Coriolis force to its right; might this give it navigational information. I published an article on this in *The Auk* 97, 99 (1980). Let me know what you think! You may have noticed the similarity between the equation for the Coriolis force $F = 2mv \times \omega$ and the equation for the Lorentz force on an electric charge moving in a magnetic field: $F = qv \times B$. The analogy can be pursued a bit further. If you rotate a coil in an electric field, a current will flow in the coil. That's electromagnetic induction, and it is the principle of an electric generator. Sometime early in the twentieth century, the American physicist Arthur Compton (that's not Denis Compton, of whom only a few of my readers will have heard, and very few indeed in North America) successfully tried an interesting experiment. He made some toroidal glass tubes, filled with water coloured with $KMnO_4$, so that he could see the water, and he rotated these tubes about a horizontal or vertical diameter, and, lo, the water flowed around the tubes, just as a current flows in a coil when it is rotated in a magnetic field. Imagine a toroidal tube set up in an east-west vertical plane at the equator. The top part of the tube is slightly further from Earth's rotation axis than the bottom part, and consequently the water near the top of tube has more angular momentum per unit mass, around Earth's axis, than the water near the bottom. Now rotate the tube through 180° about its east-west horizontal diameter. The high angular momentum fluid moves closer to Earth's rotation axis, and the low angular momentum fluid moves further from Earth's axis. Therefore, in order to conserve

angular momentum, the fluid must flow around the tube. By carrying out a series of such experiments, Compton was able, at least in principle, to measure the speed of Earth's rotation, and even his latitude, without looking out of the window, and indeed without even being aware that there was an external universe out there. You may think that this was a very difficult experiment to do, but you do it yourself every day. There are three mutually orthogonal semicircular canals inside your ear, and, every time you move your head, fluid inside these semicircular canals flows in response to the Coriolis force, and this fluid flow is detected by little nerves, which send a message to your brain to tell you of your movements and to help you to keep your balance. You have a wonderful brain, which is why understanding physics is so easy. Going back to the Lorentz force, we recall that a moving charge in a magnetic field experiences a force at right angles to its velocity.

But what is the origin of a magnetic field? Well, a magnetic field exists, for example, in the interior of a solenoid in which there is a current of moving electrons in the coil windings, and it is these circulating electrons that ultimately cause the Lorentz force on a charge in the interior of the solenoid - just as it is the galaxies in the universe revolving around our stationary Earth which are the ultimate cause of the Coriolis force on a particle moving with respect to our Earth. But there I seem to be getting into deep waters again, so perhaps it is time to move on to something easier.

4.10 The Top

We have classified solid bodies technically as symmetric, asymmetric, spherical and linear tops, according to the relative sizes of their principal moments of inertia. In this section, or at least in the title of this section, I mean "top" in the nontechnical sense of the child's toy - that is to say, a symmetric body, pointed at one end, spinning around its axis of symmetry, with the pointed end on the ground or on a table. Technically, it is a "heavy symmetric top with one point fixed." FIGURE IV.26

42 I have drawn it in figure IV.26, spinning about its symmetry axis, which makes an angle θ with the vertical. The distance between the centre of mass and the point of contact with the table is l . It has a couple of forces acting on it - its weight and the equal, opposite reaction of the table. In figure IV.27, I replace these two forces by a torque, τ , which is of magnitude $Mgl \sin \theta$. Note that, since there is an external torque acting on the system, the angular momentum vector is not fixed. FIGURE IV.27

Before getting too involved with numerous equations, let's spend a little while describing qualitatively the motion of a top, and also describing the various coordinate systems and angles we shall be discussing. First, we shall be making use of a set of space-fixed coordinates. We'll let the origin O of the coordinates be at the (fixed) point where the tip of the top touches the table. The axis Oz points vertically up to the zenith. The Ox and Oy axes are in the (horizontal) plane of the table. Their exact orientation is not very important, but let's suppose that Ox points due south, and Oy points due east. $Oxyz$ then constitutes a right-handed set. We'll also make use of a set of body-fixed axes, which I'll just refer to for the moment as 1, 2 and 3. The 3-axis is the symmetry axis of the top. The 1- and 2-axes are perpendicular to this. Their exact positions are not very important, but let's suppose that the 31-plane passes through a small ink-dot which you have marked on the side of the top, and that the 123 system constitutes a right-handed set.

43 We are going to describe the orientation of the top at some instant by means of the three Eulerian angles θ , ϕ and ψ (see figure IV.28). The symmetry axis of the top is represented by the heavy arrow, and it is tipped at an angle θ to the z-axis. I'll refer to a plane normal to the axis of symmetry as the "equator" of the top, and it is inclined at θ to the xy-plane. The ascending node of the equator on the xy-plane has an azimuth ϕ , and is the angular distance of the 1-axis from the node. The azimuth of the symmetry axis of the top is $\phi - 90^\circ$. Now let me anticipate a bit and describe the motion of the top while it is spinning and subject to the torque described above. x o y

FIGURE IV.28

44 The symmetry axis of the top is going to precess around the z-axis, at a rate that will be described as $\dot{\phi}$. Except under some conditions (which I shall eventually describe) this precessional motion is secular. That means that ϕ increases all the time - it does not oscillate to and fro. However, the symmetry axis does not remain at a constant angle with the z-axis. It oscillates, or nods, up and down between two limits. This motion is called nutation (Latin: nutare, to nod). One of our aims will be to try to find the rate of nutation $\dot{\theta}$ and to find the period and amplitude of the nutation. It may look as though the top is spinning about its axis of symmetry, but this isn't quite so. If the angular velocity vector were exactly along the axis of symmetry, it would stay there, and there would be no precession or nutation, and this cannot be while there is a torque acting on the top. An exception would be if the top were spinning vertically ($\theta = 0$), when there would be no torque acting on it. The top can in fact do that, except that, unless the top is spinning quite fast, this situation is unstable, and the top will tip away from its vertical position at the slightest perturbation. At high spin speeds, however, such motion is stable, and indeed one of our aims must be to determine the least angular speed about the symmetry axis such motion is stable. However, as mentioned, unless the top is spinning vertically, the vector ω is not directed along the symmetry axis. We'll call the three components of ω along the three body-fixed axes ω_1 , ω_2 and ω_3 , the last of these being the component of ω along the symmetry axis. One of the things we shall discover when we proceed with the analysis is that ω_3 remains constant throughout the motion. Also, you should be able to distinguish between ω_3 and $\dot{\psi}$. These are not the same, because of the motion of the node. In fact you will probably understand that $\dot{\psi} = \omega_3 - \dot{\phi} \cos \theta$. Indeed, we

have already derived the relations between the component of the angular velocity vector and the rate of change of the Eulerian angles - see equations 4.2.1, 2 and 3. We shall be making use of these relations in what follows. To analyse the motion of the top, I am going to make use of Lagrange's equations of motion for a conservative system. If you are familiar with Lagrange's equations, this will be straightforward. If you are not, you might prefer to skip this section until you have become more familiar with Lagrangian mechanics. I expect to add a chapter on Lagrangian mechanics sometime. However, I introduced Lagrange's equation briefly in section 4.4, in which Lagrange's equation of motion was given as 4.10.1 Here T is the kinetic energy of the system. P_i is the generalized force associated with the generalized coordinate q_i . If the force is a conservative force, then P can be expressed as the negative of the derivative of a potential energy function:

45 Thus we have Lagrange's equation of motion for a system of conservative forces $d(gT)/dt = 9T/9V$ 4.10.2 4.10.3 Thus, in solving problems in Lagrangian dynamics, the first line in our calculation is to write down an expression for the kinetic energy. The first line begins: " $T = \dots$ ". In the present problem, the kinetic energy is $T = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_3 \dot{\phi}^2$ 4.10.4 Here the subscripts refer to the principal axes, 3 being the symmetry axis. The Eulerian angles θ and ϕ are zenith distance and azimuth respectively of the symmetry axis with respect to laboratory fixed (space fixed) axes. The Eulerian angle ψ is measured around the symmetry axis. The components of the angular velocity are related to the rates of change of the Eulerian angles by previously derived formulas (equations 4.2.1, 2, 3), so the kinetic energy can be expressed in terms of θ , ϕ , and ψ : $T = \frac{1}{2} I_1 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\phi} \cos \theta)^2$ 4.10.5 The potential energy is $V = Mgl \cos \theta + \text{constant}$. 4.10.6 Having written down the expressions for the kinetic and potential energies in terms of the Eulerian angles, we are now in a position to apply the Lagrangian equations of motion 4.10.3 for each of the three coordinates. We'll start with the coordinate ϕ . The Lagrangian equation is 4.10.7 We see that $\partial L / \partial \phi = 0$ and $\partial L / \partial \dot{\phi}$ are each zero, so that $(-d/dt) \partial L / \partial \dot{\phi} = 0$, or has the dimensions of angular momentum, so I'll call the constant L_ϕ . derivative $\partial L / \partial \dot{\phi}$, we obtain for the Lagrangian equation in $\phi = \text{constant}$. This On evaluating the

46 $I_1 \dot{\theta}^2 \sin^2 \theta + I_3 (\dot{\psi} + \dot{\phi} \cos \theta)^2 = L_\phi^2 / I_3$ 4.10.8 I'll leave the reader to carry out exactly the same procedure with the coordinate θ . You'll quickly conclude

that $\dot{\phi} = \text{constant}$, which has the dimensions of angular momentum, so call it L_3 , and you will then arrive at the following for the Lagrangian equation in θ : $I\ddot{\theta} + \dot{\phi}^2 \cos\theta = 0$. 4.10.9 But the expression in parentheses is equal to $\dot{\phi}^2 \cos\theta$ (see equation 4.2.3, although we have already used it in equation 4.10.5), so we obtain the result that $\dot{\phi}^2 \cos\theta$, the component of the angular velocity about the symmetry axis, is constant during the motion of the top. It would probably be worth the reader's time at this point to distinguish again carefully in his or her mind the difference between $\dot{\phi}^2$ and $\dot{\phi}$. Eliminate t from equations 4.10.8 and 4.10.9: $b = L - L_3 \cos\theta$. 4.10.10 $I \sin^2\theta \dot{\theta}^2$ This equation tells us how the rate of precession varies with θ as the top nods or nutates up and down. We could also eliminate b from equations 4.10.8 and 4.10.9: $\dot{\phi}^2 (\frac{1}{2} - \frac{1}{2} \cos^2\theta) \cos\theta = \dot{\theta}^2$. 4.10.11 $I \sin\theta \dot{\theta}^2$ The Lagrangian equation in θ is a little more complicated, but we can obtain a third equation of motion from the constancy of the total energy: $\frac{1}{2} I (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I \dot{\phi}^2 \cos^2\theta + Mgl \cos\theta = E$. 4.10.12 We can eliminate b and t from this, using equations 4.10.10 and 11, to obtain an equation in θ and the time only. After a little algebra, we obtain $6I^2 = A - B \cos\theta - L - L_3 \cos\theta$: $-r / ' 4.10.13$

47 where 4.10.14 and $B = 2Mgl$. 4.10.15 The turning points in the θ -motion (i.e. the nutation) occur where (after some algebra! $c = \cos\theta'$ where $t) = 0$. This results - but quite straightforward. all the same) in a cubic equation in a $\theta + ac + agc: + Bc^3 = 0$. 4.10.16 $I^2 \sin^2\theta \dot{\theta}^2 + a\theta = A - L - \frac{1}{2} E L^3 L$, 4.10.17 $2LL^3 2LL^3 2Mgl a - B - 4.10.18$ and Now equation 4.10.16 is a cubic equation in $\cos\theta$ and it has either one real root or three real roots, and in the latter case two of them or all three might be equal. We must also bear in mind that θ is real only if $\cos\theta$ is in the range -1 to +1. We are trying to find the nutation limits, so we are hoping that we will find two and only two real values of θ . (If the tip of the top were poised on top of a point - e.g. if it were poised on top of the Eiffel Tower, rather than on a horizontal table - you could have $\theta > 90^\circ$.) To try and understand this better, I constructed in my mind a top somewhat similar in shape to the one depicted in figures IV.25 and 26, about 4 cm diameter, 7 cm high, made of brass. For the particular shape and dimensions that I imagined, it worked out to have the following parameters, rounded off to two significant figures: $M = 0.53 \text{ kg}$ $I = 0.044 \text{ m}^2 \text{ kgm}^2$ $I_3 = 9.8 \times 10^{-5} \text{ kgm}^2$ I thought I'd spin the top so that $\dot{\phi}^2$ (which, as we have seen, remains constant throughout the motion) is $250 \text{ rad}^2/\text{s}^2$, and I'd start the top at rest ($b = \dot{\theta} = 0$) at $\theta = 30^\circ$ and then let go. Presumably it

would then immediately start to fall, and 30° would then be the upper bound to the nutation. We want to see how far it will fall before nodding upwards again. With $\dot{\theta} = 250 \text{ rad/s}$ we find, from equation 4.10.9 that $L^3 = 2.45 \times 10^{-2} \text{ J s}$.

48 Also, I am assuming that $(b = 0 \text{ when } \theta = 30^\circ)$, and equation 4.10.10 tells us that $L = 2.121762 \times 10^{-2} \text{ J s}$. Then with $g = 9.8 \text{ m/s}^2$, we have, from equation 4.10.15, $B = 2.688659 \times 10^3 \text{ s}^2$. My initial conditions are that $(b$ and t are each zero when $\theta = 30^\circ$), and equations 4.10.10 and 4.10.13 between them tell us that $\dot{\theta} = B \cos \theta$, so that $2.328447 \times 10^3 \text{ s}^2$. From equations 4.10.17, 18 and 19 we now have $a_0 = -1.324898 \times 10^4 \text{ S}^2$, $a_1 = +3.328586 \times 10^4 \text{ s}^2$, $a_2 = -2.309834 \times 10^4 \text{ s}^2$ and we already have $B = 2.688659 \times 10^3 \text{ s}^2$. The "sign rule" for polynomial equations, if you are familiar with it, tells us that there are no negative real roots (i.e. no solutions with $\theta > 90^\circ$), and indeed if we solve the cubic equation 4.10.16 we obtain $c = 0.824596, 0.866025, 6.900406$. The second of these corresponds to $\theta = 30^\circ$, which we already knew must be a solution. Indeed we could have divided equation 4.10.16 by $c - \cos \theta$ to obtain a quadratic equation for the remaining two roots, but it is perhaps best to solve the cubic equation as it is, in order to verify that $\cos 30^\circ$ is indeed a solution, thus providing a check on the arithmetic. The third solution does not give us a real θ (we were rather hoping this would happen). The second solution is the lower limit of the nutation, corresponding to $\theta = 34^\circ 27'$. It can happen if all conditions are just right, that the two real solutions for θ are equal, in which case there is no nutation, and we have true regular precession.

49 Generally, however, the top will nutate between two values of θ . Let us call these two values θ_1 and θ_2 , θ_1 being the smaller (more vertical) of the two. I'll refer to $\theta = \theta_1$ as the "upper bound" of the motion, even though $\theta_1 < 3$, since this corresponds to the more vertical position of the top. We have looked a little at the motion in θ ; now let's look at the motion in ϕ , starting with equation 4.10.10: $L - L^3 \cos \theta = I \sin^2 \theta \dot{\phi}^2$. 4.10.10 If the initial conditions are such that $L > L^3 \cos \theta_1$ (and therefore always greater than $\cos 0$) is always positive. The motion is then something like I try to illustrate in figure IV.29. FIGURE IV.29 This motion corresponds to an initial condition in which you give the top an initial push in the forward direction as indicated by the little blue arrow. If the initial conditions are such that $\cos \theta_1 > L/L^3 > \cos \theta_2$, the sign

of b is different at the upper and lower bounds. Thus is illustrated in figure IV.30

50 FIGURE IV.30 This motion would arise if you were initially to give a little backward push before letting go of the top, as indicated by the little blue arrow. If the initial conditions are such that $L = L_3 \cos \theta_0$ or, then $\dot{\theta}$ and b are each zero at the upper bound of the nutation, and this was the situation in our numerical example. It corresponds to just letting the top drop when you let it go, without giving it either a forward or a backward push. This is illustrated in figure IV.31. FIGURE IV.31 As we discovered while doing our numerical example, $\dot{\theta} = 0$ when $\theta = \theta_0$ lead, in this third type of motion, to the initial conditions $L = L_3 \cos \theta_0$ 4.10.20

51 and $A = B \cos \theta_0$. 4.10.21 In that case equation 4.10.13 becomes $0 = \frac{1}{2} L^2 \dot{\theta}^2 - I \sin^2 \theta - \frac{Mg}{\cos \theta} - \frac{L^2 \dot{\phi}^2}{2} - \frac{L^2 \dot{\psi}^2}{2}$ 4.10.22 The lower bound to the nutation (i.e. how far the top falls) is found by putting $\dot{\theta} = 0$ when $\theta = \theta_3$. This gives the following quadratic equation for $\cos \theta_3$: $2 \cos^2 \theta_3 - 7.725002 \cos \theta_3 + 5.690048 = 0$, 4.10.23 which, naturally, has the same two solutions as we obtained when we solved the cubic equation, namely 0.824596 and 6.900406. Recalling the definition of B (equation 4.10.15), we see that equation 4.10.23 can be written $\cos \theta_3 - \cos \theta_0 - 2Mg/I \sin \theta_3 = 0$, 4.10.24 from which we see that the greater L_3 , the smaller the difference between θ_0 and θ_3 - i.e. the smaller the amplitude of the nutation. Equation 4.10.12, with the help of equations 4.10.10 and 11, can be written: $E - \frac{1}{2} L^2 \dot{\theta}^2 - \frac{1}{2} (L \csc \theta - L \cot \theta)^2 + Mg/\cos \theta = 0$ 4.10.25 2.12 2I The left hand side is the total energy minus the spin and nutation kinetic energies. Thus the right hand side represents the effective potential energy $V_{\text{eff}}(\theta)$ referred to a reference frame that is co-rotating with the precession. The term $Mg/\cos \theta$ needs no explanation. The negative of the derivative of the first term on the right hand side would be the "fictitious" force that "exists" in the corotating reference frame. The effective potential energy $V_{\text{eff}}(\theta)$ is given by $2IMg/I \cos \theta - \frac{1}{2} L^2 \dot{\theta}^2 - \frac{1}{2} (L \csc \theta - L \cot \theta)^2 + Mg/\cos \theta$: + 4.10.26

52 I draw $V_{\text{eff}}(\theta)$ in figures IV.32 and 33 using the values that we used in our numerical example - that is: $V_{\text{eff}}(\theta) = 1.324081(\csc \theta - 1.154701 \cot \theta)^2 + 0.228536 \cos \theta$ joules. 4.10.27 Figure 32 is plotted up to 90° (although as

mentioned earlier one could go further than this if the top were not spinning on a horizontal table), and Figure 33 is a close look close to the minimum. One can see that if $E - L^2/(2I) = 0.1979$ the effective potential energy (which cannot go higher than this, and reaches this value only when $\theta = 0$), the nutation limits are between 30° and $34^\circ 24'$. For a given L , for a larger total energy, the nutation limits are correspondingly wider. But for a given total energy, the larger the component L_3 of the angular momentum is, the lower will be the horizontal line and the narrower the nutation limits. If $E - L^2/(2I)$ is equal to the minimum value of $V_e(\theta)$ there is only one solution for θ , and we have true regular precession.

4.5 4 3.5 2.5 2 1.5 0.5
 FIGURE IV. 32 0 1 1 1 1 1 1 0 10 20 30 40 50 60 70 80 90 8, degrees

53 0.225 0.22 0.215 0.21 0.205 0.2 FIGURE IV. 33 0.195 25 30 35 40 0, degrees We can find the rate of true regular precession quite simply as follows - and this is often done in introductory books. 33 FIGURE IV.34 In figure IV.34, the vector L represents the angular momentum at some time, and in a time interval Δt later the change in the angular momentum is ΔL . The angular momentum

54 is changing because of the external torque, which is a horizontal vector of magnitude $Mg \sin \theta$ (remind yourself from figure XV.26 and 27). The rate of change of angular momentum is given by $\dot{L} = \tau$. In time Δt the tip of the vector L moves through a "distance" $c \Delta t$. Denote by Ω precessional angular velocity (the magnitude of which we have hitherto called $\dot{\phi}$). The tip of the angular momentum vector is moving in a small circle of radius $L \sin \theta$. We therefore see that $c = \Omega L \sin \theta$. Further, $\dot{L}$ is perpendicular to both Ω and L . Therefore, in vector notation, $\dot{L} = \Omega \times L$. 4.10.29 Not that the magnitude of $\dot{L}$ is $Mg \sin \theta$ and the magnitude of $\Omega \times L$ is $\Omega L \sin \theta$, so that the rate of precession is $\Omega = Mg / L \cos \theta$. 4.10.30 Ω is independent of θ . One can continue to analyse the motion of a top almost indefinitely, but there are two special cases that are perhaps worth noting and which I shall describe. Special Case I. $L = L_3$. In this case, equation 4.10.27 becomes $V_e(\theta) = Mg l - C(\csc \theta - \cot \theta)^2 + \cos \theta$, 4.10.31 where $C = L^2 / 2I$. 4.10.32 $2Mg l$ It may be rather unlikely that $L = L_3$ exactly, but this case is of interest partly because it is exceptional in that $V_e(\theta)$ does not go to infinity; in fact $V_e(\theta) = Mg l$ whatever the value of C . Try substituting $\theta = 0$ in equation 4.10.32 and see what you get! The right hand side is indeed 1, but you may have to work a little to get there. The

other reason why this case is of interest is that it makes a useful introduction to case II, which is not impossibly unlikely, namely that L is approximately equal to L_3 , which leads to motion of some interest. In figure IV.35, I draw $Ve(O)/Mgl$ for several different C .

55 2 1.8 1.6 . 1.4 1.2 $C=10$ FIGURE IV. 2 0.8 0 10 20 30 40 50 60 70 80 90
 0, degrees From the graphs, it looks as though, if $C \geq 2$, there is one equilibrium position, it is at $\theta = 0$ (i.e. the top is vertical), and the equilibrium is stable. If $C < 2$, there are two equilibrium positions: the vertical position is unstable, and the other equilibrium position is stable. Thus if the top is spinning fast (look at the definition of C - equation 4.10.32), it can spin in the vertical position only (a "sleeping top"), but, as the top slows down owing to friction and air resistance, the vertical position will become unstable, and the top will fall down to a positive value of θ . These deductions are correct, for $dVe/d\theta$ results in $2C(1 - \cos \theta) = \sin \theta$ 4.10.33 One solution is $\theta = 0$, and a second differentiation will show that this is stable or unstable according to whether C is greater than or less than 2, although the second differentiation is slightly tedious, and it can be avoided. We can also note that $1 - \cos \theta$ is a common factor of the two sides of equation 4.10.33, and it can be divided out to yield a cubic equation in $\cos \theta$: $2C - 1 - (2C+1)\cos\theta - \cos^2\theta - \cos^3\theta = 0$, 4.10.34

56 which could be solved to find the second equilibrium point - but that again is slightly tedious. A less tedious way might be to take the square root of each side of equation 4.10.33: $2C/(1 - \cos\theta) = 1/\sin\theta$ 4.10.35 and then divide by the common factor $(1 - \cos \theta)$ to obtain $\cos\theta: 2C/(1 + \cos\theta) = 1$, 4.10.36 which gives a real θ only if $C > 2$. Note also, if $C = 3$ $\theta = 90^\circ$. Special Case I. $L = L_3$. In other words, L and L_3 are not very different. several different C , for $L/L_3 = 1.01$. In figure IX.36 I draw $Ve(O)/Mgl$ for 2 1.8 1.6 1.2 0.8 0 FIGURE IV. 36 $C=10$ 10 20 30 40 50 60 70 80 90 θ , degrees We see that for quite a large range of C greater than 2 the stable equilibrium position is close to vertical. Even though the curve for $C = 2$ has a very broad minimum, the actual minimum is a little less than 17° . (I haven't worked out the exact position - I'll leave that to the reader.)

CHAPTER 5 COLLISIONS 5.1 Introduction In this chapter on collisions, we shall have occasion to distinguish between elastic and inelastic collisions. An

elastic collision is one in which there is no loss of translational kinetic energy. That is, not only must no translational kinetic energy be degraded into heat, but none of it may be converted to vibrational or rotational kinetic energy. It is well known, for example, that if a ball makes a glancing (i.e. not head-on) elastic collision with another ball of the same mass, initially stationary, then after collision the two balls will move off at right angles to each other. But this is so only if the balls are smooth. If they are rough, after collision the balls will be spinning, so this result - and any other results that assume no loss of translational kinetic energy - will not be valid. When molecules collide, they may be set into rotational and vibrational motion, and in that case the collision will not be elastic in the sense in which we are using the term. If two atoms collide, one (or both) may be raised to an excited electronic level. Some of the translational kinetic energy has then been converted to potential energy. If the excited atom subsequently drops down to a lower level, that energy is radiated away and lost from the system. Superelastic collisions are also possible. If one atom, before collision, is in an excited electronic state, on collision it may make a radiationless downwards transition, and the potential energy released is then converted to translational kinetic energy, so the collision is superelastic. None of this is intended to mean that elastic collisions are impossible or even rare. In the case of collisions involving macroscopic bodies, such as smooth, hard billiard balls, collisions may not be 100% elastic, but they may be close to it. In the case of low-energy (low temperature) collisions between atoms, there need be no excitation to excited levels, in which case the collision will be elastic. Some subatomic particles, in particular leptons (of which the electron is the best-known example), are believed to have no internal degrees of freedom, and therefore collisions between them are necessarily elastic. In laying out the principles involved in collisions between particles, we need not suppose that the particles actually "bang into" - i.e. touch - each other. For example most of the principles that we shall be describing apply equally to collisions between balls that "bang into" each other and to phenomena such as Rutherford scattering, in which an alpha particle is deviated from its path by a gold nucleus without actually "touching" it. Of course, if you think about it at an atomic level, when two billiard balls collide, the atoms don't actually "touch" each other; they are repelled from each other by electromagnetic forces, just as the alpha particle and the gold nucleus repelled each other in the Rutherford-Geiger-Marsden experiment. The theory of collisions is used

a great deal, of course, in the study of high-energy collisions between particles in particle physics. Bear in mind, however, that in "atom-smashing" experiments with modern huge particle accelerators, or even in relatively mild collisions such as Compton scattering of x-rays, the particles involved are moving at speeds that are not negligible compared with the speed of light, and therefore relativistic mechanics is needed for a proper analysis. In this chapter, collisions are treated entirely from a nonrelativistic point of view.

5.2 Bouncing Balls When a ball is dropped to the ground, one of four things may happen: 1. It may rebound with exactly the same speed as the speed at which it hit the ground. This is an elastic collision. 2. It may come to a complete rest, for example if it were a ball of soft putty. I shall call this a completely inelastic collision. 3. It may bounce back, but with a reduced speed. For want of a better term I shall refer to this as a somewhat inelastic collision. If there happens to be a little heap of gunpowder lying on the table where the ball hits it, it may bounce back with a faster speed than it had immediately before collision. That would be a superelastic collision. The ratio speed after collision is called the coefficient of restitution, for which I shall use the speed before collision symbol e . The coefficient is 1 for an elastic collision, less than 1 for an inelastic collision, zero for a completely inelastic collision, and greater than 1 for a superelastic collision. The ratio of kinetic energy (after) to kinetic energy (before) is evidently, in this situation, e^2 . 5.2.1 If a ball falls on to a table from a height h_0 , it will take a time $t_0 = 2h_0/g$ to fall. If the collision is somewhat inelastic it will then rise to a height $h = e^2 h_0$ and it will take a time $e t_0$ to reach height h . Then it will fall again, and bounce again, this time to a lesser height. And, if the coefficient of restitution remains the same, it will continue to do this for an infinite number of bounces. After a billion bounces, there is still an infinite number of bounces yet to come. The total distance travelled is $h = h_0 + 2h_0(1 + e^2 + e^4 + e^6 + \dots)$ 5.2.1 and the time taken is $t = t_0 + 2t_0(1 + e + e^2 + e^3 + \dots)$. 5.2.2 These are geometric series, and their sums are $h (1 - e^2)$ 5.2.3 which is independent of g (i.e. of the planet on which this experiment is performed), and $t (1 + e)$ 5.2.4

For example, suppose $h_0 = 1$ m, $e = 0.5$, $g = 9.8$ m s⁻², then the ball comes to rest in 1.36 s after having travelled 1.67 m after an infinite number of bounces. Discuss. (E.g. Does it ever stop bouncing, given that, after every

bounce, there is still an infinite number yet to come; yet after 1.36 seconds it is no longer bouncing...?)

5.3 Head-on Collision of a Moving Sphere with an Initially Stationary Sphere

Before: After: FIGURE V. 1

The coefficient of restitution is relative speed of recession after collision $e = \frac{v_2 - v_1}{u_1 - u_2}$

5.3.1 relative speed of approach before collision We suppose that the two masses m and m_2 , the initial speed u , and the coefficient of restitution e are known; we wish to find v and v_2 . We evidently need two equations. Since there are no external forces on the system, the linear momentum of the system is conserved: $mu = mv + m_2v_2$

5.3.2 The second equation will be the restitution equation (equation 5.3.1): $v_2 - v_1 = eu$

5.3.3 These two equations can be solved to yield $v = \frac{m - m_2e}{m + m_2}u$

5.3.4 $v_2 = \frac{m(1 + e)}{m + m_2}u$

and 5.3.5 The relation between the kinetic energy loss and the coefficient of restitution isn't quite as simple as in section 5.2. Exercise. Show that kinetic energy (after) $= \frac{1}{2}mv^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}(m + m_2e^2)u^2$

5.3.6 $\frac{m}{m + m_2}$ If $m_2 = \infty$ (as in section 5.2), this becomes just e^2 . If $e = 1$, it becomes unity, so all is well. If $m_1 \ll m_2$ (Ping-pong ball collides with cannon ball), $v_1 = -u$, $v_2 = 0$. If $m_1 = m_2$ (Ping-pong ball collides with ping-pong ball), $v_1 = 0$, $v_2 = u$. If $m_1 \gg m_2$ (Cannon ball collides with ping-pong ball), $v_1 = u$, $v_2 = 2u$.

5.4 Oblique Collisions

In figure V.2 I show two balls just before collision, and just after collision. The horizontal line is the line joining the centres - for short, the "line of centres". We suppose that we know the velocity (speed and direction) of each ball before collision, and the coefficient of restitution. The direction of motion is to be described by the angle that the velocity vector makes with the line of centres. We want to find the velocities (speed and direction) of each ball after collision. That is, we want to find four quantities, and therefore we need four equations. These equations are as follows.

U2 Before: m Line of centres After: U1

FIGURE V.2 Line of centres

There are no external forces on the system along the line of centres. Therefore the component of momentum of the system along the line of centres is conserved: $mv \cos \theta + m_2U_2 \cos \theta_2 = mu \cos \theta + m_2u_2 \cos \theta_2$

5.4.1 If we assume that the balls are smooth - i.e. that there are no forces perpendicular to the line of centres and the balls are not set into rotation, then the component of the momentum of each ball separately perpendicular to the line of centres is conserved: $mv \sin \theta = u \sin \theta$

5.4.2 $v^2 \sin^2 \theta = u^2 \sin^2 \theta$

c 2 . 5.4.3 The last of the four equations is the restitution equation relative speed of recession along the line of centres after collision $e = \text{relative speed of approach along the line of centres before collision}$ That is, $v_2 \cos \theta_2 - v_1 \cos \theta_1 = e(u_1 \cos \theta_1 - u_2 \cos \theta_2)$. 5.4.4 Example. Suppose $m_1 = 3 \text{ kg}$, $m_2 = 2 \text{ kg}$, $u_1 = 40 \text{ m s}^{-1}$, $u_2 = 15 \text{ m s}^{-1}$, $\theta_1 = 10^\circ$, $\theta_2 = 70^\circ$, $e = 0.8$ Find v_1 , v_2 , θ_1 , θ_2 . Answers: $v_1 = 16.28 \text{ m s}^{-1}$, $v_2 = 25.15 \text{ m s}^{-1}$, $\theta_1 = 44.43^\circ$, $\theta_2 = 30^\circ$ Example. Suppose $m_1 = 2 \text{ kg}$, $m_2 = 3 \text{ kg}$, $u_1 = 12 \text{ m s}^{-1}$, $u_2 = 20 \text{ m s}^{-1}$, $\theta_1 = 20^\circ$, $\theta_2 = 50^\circ$, $e = 0.6418$ Find v_1 , v_2 , θ_1 , θ_2 . Answers: $v_1 = 10.50 \text{ m s}^{-1}$, $v_2 = 15.71 \text{ m s}^{-1}$, $\theta_1 = 23^\circ$, $\theta_2 = 0^\circ$ Problem. If $u_2 = 0$, and $e = 1$ and $m_1 = m_2$, show that $\theta_1 = 90^\circ$ and $\theta_2 = 0^\circ$.

5.5 Oblique (Glancing) Elastic Collisions, Alternative Treatment Before:

After: $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$ FIGURE V.3 m_2 In figure V.3, unlike figure V.2, the horizontal line is not intended to represent the line of centres. Rather, it is the direction of the initial velocity of m_1 , and m_2 is initially at rest. The second mass m_2 is slightly off the line of the velocity of m_1 . I am assuming that the collision is elastic, so that $e = 1$. In the "before" part of the figure, I have indicated, as well as the two masses, the position and velocity V of the centre of mass C . The velocity of C remains constant, because there are no external forces on the system. I have not drawn C in the "after" part of the figure, because it would get a little in the way. Think about where it is. Figure V.3 shows the situation in "laboratory space". (Later, we'll look at the situation referred to a reference frame in which C is at rest - "centre of mass space".) The angle θ is the angle through which m_1 has been scattered (the "scattering angle"). I have indicated in the figure how it is related to the θ and ϕ of section 5.3. Note that m_2 (initially stationary) scoots off along the line of centres. The following two equations express the constancy of linear momentum of the system. $(m_1 + m_2)V = m_1 u_1 \cos \theta + m_2 v_2 \cos \phi$. 5.5.1 I'm going to draw, in figure V.4, the situation "close-up", so that you can see the geometry more clearly. Note that the distance b is called the impact parameter. It is the distance by which the two centres would have missed each other had the first particle not been scattered. In figure V.5, I draw the situation in centre of mass space, in which the centre of mass C is stationary. In this reference frame, I just have to subtract V from all the velocities. Note that in centre of mass space the speeds of the particles are unaltered by the collision. In centre of mass space, m_1 is scattered through an angle θ' , and I am going to find a relation between θ' , θ and the mass ratio m_2/m_1 . I shall start

with the profound statement that $\tan \theta = v \sin \theta / v \cos \theta$

Before After $b = (R + R_2 \sin \theta) / R_2 \sin \theta$.. $R_2 \sin \theta = R_2 \sin \theta$ FIGURE V.4 Before: After: m_1
FIGURE V.5

Now $v \sin \theta$ is the y-component of the final velocity of m_1 in laboratory space. The y-component of the final velocity of m_1 in centre of mass space is $u' \sin \theta'$, and these two are equal, since the y- component of the motion is unaffected by the change of reference frame. Therefore $u' \sin \theta' \tan \theta = v \sin \theta$ - 5.5.3
Therefore $v \sin \theta = u' \sin \theta'$. The x-components of the "before" and "after" velocities of m_1 are related by $v \cos \theta = u' \cos \theta' + V$. Substitute equations 5.5.4 and 5.5.6 into equation 5.5.2 to obtain $\sin \theta' \tan \theta = \cos \theta' + V/u'$ 5.5.4 5.5.6 5.5.7 But $(m+m_2)V = mu = m(u'+V)$, 5.5.8 $V = mu/(m+m_2)$ from which $-- = ' m_2$ 5.5.9 On substituting this into equation 5.5.7, we obtain the relation we sought: $\sin \theta' \tan \theta = \cos \theta' + m_2/m$ 5.5.10 This relation is illustrated in figure V.6 for several mass ratios. Let us try to interpret the figure. For $m_2 > m_1$, any scattering angle, forward or backward, is possible, but for $m_2 < m_1$, backward scattering is not possible, and forward scattering is possible only up to a maximum. This is only to be expected. Thus for an impact parameter of zero or of $R + R_2$, and $m_2 < m_1$, the scattering angle θ must be zero, and therefore for intermediate impact parameters it must go through a maximum. This would be clearer if we could plot the scattering angle versus the impact parameter, and indeed that is something that we shall try to do. In the meantime it is easy to show, by differentiation of equation 5.5.10 (do it!), that the maximum scattering angle is $\sin g$, where $g = m_2/m$. That is, if the scattered particle is very massive compared with the scattering particle, the maximum scattering angle is small - just to be expected.

180 160 I 140 FIGURE V.6 120 , $LL = m_2/m_1 = 100$ 80 60 40 0.5 0.25 20
9/10 0 0 20 40 60 80 100 120 140 160 180 i want to do two things now - one, to calculate the scattering angle θ as a function of impact parameter, and two, to calculate v/u as a function of scattering angle. I'm going to start with equations 5.4.1,2 and 4, except for the following. i'll assume $e = 1$ (elastic collision), and $u_2 = 0$ (m_2 is initially stationary), and $\theta_2 = 0$ (since m_2 is initially stationary, it must move along the line of centres after collision). Since i want to try to calculate the scattering angle, i'll write $\theta + \theta_2$ for θ (see figure V.4). i'm also going to write r_1 , r_2 and g for the dimensionless ratios

u/u , u^2/u and m^2/m respectively. With those small changes, equations 5.4.1, 2 and 4 become $q \cos(\theta + \phi) + p t = \cos \theta$, 5.5.11 $q \sin(\theta + \phi) = \sin \theta$, 5.5.12 $r - q \cos(\theta + \phi) = \cos \theta$. 5.5.13 Eliminate r^2 from equations 5.5.11 and 5.5.13 to obtain $q \cos(\theta + \phi) = M \cos \theta$, 5.5.14

where $M = 1 - g = m - m^2$ 5.5.15 $1 + \frac{1}{2} m + \frac{1}{2} m^2$ If we now eliminate ϕ from equations 5.5.12 and 14, we obtain the relation between v/u and the scattering angle, which was the second of our two aims above. The elimination is easily done as follows. Expand $\sin$ and $\cos$ of $\theta + \phi$ in the two equations, divide both sides of each equation by $\cos \phi$ and eliminate $\tan \phi$ between the two equations. The result is $2 - q(1+M)\cos\theta + M = 0$. 5.5.16 We'll have a look at this equation in a moment, but in the meantime, instead of eliminating ϕ from equations 5.5.12 and 14, let's eliminate r . This will give us a relation between the scattering angle θ and ϕ , and, since ϕ is closely related to the impact parameter (see figure V.4) this will achieve the first of our aims, namely to find the scattering angle as a function of the impact parameter. If you do the algebra, you should find that the relation between θ and ϕ is $\tan \phi = \frac{a(1-M)}{a+M}$ 5.5.17 where $a = \tan \theta$. 5.5.17a,b Now let $b' = \frac{R}{R^2}$ and from figure V.5 we see that $b' = \sin \theta$. 5.5.18 On elimination of ϕ from equations 5.5.17 and 5.5.19, we obtain the required relation between scattering angle θ and (dimensionless) impact parameter b' : $\tan \theta = \frac{2gb'x}{b'^2 - 1 - \frac{1}{2} + 2[tb'^2}$ 5.5.20 This relation is shown in figure V.7. The values of the mass ratio $g (= m^2/m)$ are (from the 1 1 1 9 1 - 2 4 8 and (dashed) oz. This figure is perhaps slightly easier to lowest up) 8,4, 2,0 interpret than figure V.6. One can see that for $g > 1$, any scattering angle is possible, but for $g < 1$, the scattering angle has a maximum possible value, less than 90° ♦, and the scattering angle is zero for $b' = 0$ or 1. Exercise. We saw, by differentiation of equation 5.5.10, that the maximum scattering angle was $\sin g$. Now show the same thing by differentiation of equation 5.5.20. (This is not so easy, is it?) Show that the scattering angle is greatest for an impact parameter of

5.5.21 You will notice that, for $b' = 0$ (head-on collision) the scattering angle changes abruptly from 0 to 180° ♦ as the mass ratio changes from less than 1 to more than 1. No problem there. But if the mass ratio is exactly 1 (not the tiniest bit less or the tiniest bit more) the scattering angle is apparently 90° ♦. This may cause some puzzlement until it is realized that for a head-on

collision with $p = 1$ the first sphere comes to a dead halt. The case of $p = \infty$ (second sphere immovable) is of some interest. It is easy in that case to calculate how the scattering angle varies with impact parameter for an elastic collision, merely by requiring the scattered sphere to obey the law of reflection, and without any reference to equation 5.5.20. 180 160 140 120 100 80 60 40 20

FIGURE V.7 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 b' Easy Exercise. Without any reference to equation 5.5.20, show that, immovable, the scattering angle is related to the impact parameter by if the second ball is $0 = 180^\circ - 2\sin b'$. 5.5.22

Not-so-easy exercise. Show that, in the limit as it $\rightarrow \infty$, equation 5.5.20 approaches equation 5.5.22. In any case, the limiting case as the second sphere becomes immovable is shown as a dashed curve in figure V.7.

Exercise of Intermediate Difficulty. The mass ratio m_2/m_1 is 0.9, and the scattering angle is 50° . What was the impact parameter? Answers $b' = 0.07270$ or 0.58540 . We have now dealt with the direction of motion of m after scattering as a function of impact parameter. We should now look at the speed of m after collision, and this takes us back to equation 5.5.16, which gives is the speed ($v = v/u$) as a function of scattering angle θ . it is quadratic in r , so, for a given scattering angle there are two possible speeds - which is not surprising, because a given scattering angle can arise from two different impact parameters, as we have just found out. We can conveniently show the relation between r and θ simply by plotting the equation in polar coordinates. i'll re-write the equation here for easy reference: $52 - q(1+M)\cos\theta + M = 0$.

5.5.16 Here, $M = 1 - m_2/m_1$ it $m_1 - m_2$ $1 + m_2/m_1$ it $m_1 + m_2$ --, but i want to write the equation in terms of the mass fractions m_1/m_2 it q and q^2 5.5.23a,b $m_1 + m_2$ $1 + m_2/m_1$ it $m_1 + m_2$ $1 + m_2/m_1$ If you work at this for a short while, you will find that equation 5.5.16 becomes $q^2 + q - 2q \cos\theta = q^2$, 5.5.24 and one is then overcome with an overwhelming desire to draw a triangle: rl 611 FIGURE V.8

For a given mass ratio, the locus of r (the speed) versus θ (the scattering angle) is such that q and q^2 are constant - in other words, it is a circle: FIGURE V.9 FIGURE V. 10

One can imagine that the first particle comes in from the left at speed u and the collision takes place at the asterisk, and, after collision, it is moving at a

speed r times u in a direction θ , the magnitude of its velocity vector being determined by where the vector intersects the circle (in two possible places) given by equation 5.5.24. The maximum scattering angle corresponds to a velocity vector that is tangent to the circle. If the asterisk is the pole (origin) of the polar coordinates, the centre of the circle is at a distance q from the pole, and its radius is q^2 . Figure V. 10 shows the circles corresponding to several mass ratios. The figure graphically illustrates the relation between u , t , θ and it . You can see, for example, that if $it > 1$, scattering through any angle is possible, and the relation between t and θ is unique; but if $it < 1$, only forward scattering is possible, up to a maximum θ , and, for a given θ , there are two solutions for t . This deals with what happens to the sphere m . We can now turn our attention to m_2 . Starting from equations 5.5.11, 12 and 13, we are going to want to eliminate r and θ - indeed anything that pertains to the sphere m . If you refer to figure V.4 you will see that, after collision, m_2 scoots off at an angle θ' to the original direction of motion of m . Therefore I think it is of interest to find a relation between $r^2 (=t^2/u)$ and θ' . If we succeed in doing this, it means that we can also find a relation, if we want it, between r^2 and the impact parameter, since $b' = \sin \theta'$. It is easy to eliminate r from equations 5.5.12 and 13, and then you can get $\tan(\theta + \theta')$ from equation 5.5.14, and hence get the required relation: $\cos \theta [1 + \frac{1}{g}] = \frac{r^2}{1 + g}$. I'll draw this relation as a polar graph, r^2 versus θ' , in figure V.11. I'll leave the reader to work out and draw the relation between r^2 and b' if he or she wishes. Equation V. 11 is the polar equation to a circle of radius $1/(1 + g)$. Example. Suppose the mass ratio $it = m_1/m_2 = 0.5$ and the scattering angle is $\theta = 20^\circ$. Equation 5.5.16 or figure V.10 will show that $r = 0.8696$ or 0.3833 . Equation 5.5.17 will show that $\theta' = 58.4^\circ$ or 11.6° . And equation 5.5.25 or figure V. 11 will show that $r^2 = 0.6983$ or 1.306 . I'll leave it to the reader to determine which alternative values of r , r^2 and θ' go together.

i 1 FIGURE V. 11

1 CHAPTER 6 MOTION IN A RESISTING MEDIUM 1. Introduction In studying the motion of a body in a resisting medium, we assume that the resistive force on a body, and hence its deceleration, is some function of its speed. Such resistive forces are not generally conservative, and kinetic energy is usually dissipated as heat. For simple theoretical studies one can assume a simple force law, such as the resistive force is proportional to the speed, or to

the square of the speed, or to some function that we can conveniently handle mathematically. For slow, laminar, nonturbulent motion through a viscous fluid, the resistance is indeed simply proportional to the speed, as can be shown at least by dimensional arguments. One thinks, for example, of Stokes's Law for the motion of a sphere through a viscous fluid. For faster motion, when laminar flow breaks up and the flow becomes turbulent, a resistive force that is proportional to the square of the speed may represent the actual physical situation better.

2. Uniformly Accelerated Motion. Before studying motion in a resisting medium, a brief review of uniformly accelerating motion might be in order. That is, motion in which the resistance is zero. Any formulas that we develop for motion in a resisting medium must go to the formulas for uniformly accelerated motion as the resistance approaches zero. One may imagine a situation in which a body starts with speed v_0 and then accelerates at a rate a . One may ask three questions: How fast is it moving after time t ? How far has it moved in time t ? How fast is it moving after it has covered a distance x ? The answers to these questions are well known to any student of physics:

6.2.1 $x = v_0 t + \frac{1}{2} a t^2$

6.2.2 $v^2 = v_0^2 + 2ax$

6.2.3 Since the acceleration is uniform, there is no need to use calculus to derive these. The first follows immediately from the meaning of acceleration. Distance travelled is the area under a speed: time graph. Figure VI.1 shows a speed: time graph for constant acceleration, and equation 6.2.2 is obvious from a glance at the graph. Equation 6.2.3 can be obtained by elimination of t between equations 6.2.1 and 6.2.2. (It can also be deduced from energy considerations, though that is rather putting the cart before the horse.)

speed $0t$ FIGURE IV. 1 at time Nevertheless, although calculus is not necessary, it is instructive to see how calculus can be used to analyse uniformly accelerated motion, since calculus will be necessary in less simple situations. We shall be using calculus to answer the three questions posed earlier in the section. For uniformly accelerated motion, the equation of motion is $a = dv/dt$.

6.2.4 To answer the first question, we write a as dv/dt , and then the integral (with initial condition $x=0$ when $t=0$) is $v = v_0 + at$.

6.2.5 $x = v_0 t + \frac{1}{2} a t^2$

6.2.6 This is the second time integral. To obtain the answer to the third question, which will be called the space integral, we must remember to write a as $v dv/dx$. Thus the equation of motion (equation 6.2.4) is $v dv = a dx$.

6.2.7 $\int v dv = \int a dx$ This is the first time integral. Next, we write v as dx/dt and integrate

again with respect to time, to get

When this is integrated with respect to x (with initial condition $v = v_0$ when $x = 0$) we obtain $v^2 = v_0^2 + 2ax$. 6.2.8 This is the space integral. Examples. Here are a few quick examples of problems in uniformly accelerated motion. It is probably a good idea to work in algebra and obtain algebraic solutions to each problem. That is, even if you are told that the initial speed is 15 m s^{-1} , call it v_0 , or, if you are told that the height is 900 feet, call it h . You will probably find it helpful to sketch graphs either of distance versus time or speed versus time in most of the problems. One last little hint: Remember that the two solutions of a quadratic equation are equal if $b^2 = 4ac$.

1. A body is dropped from rest. The last third of the distance before it hits the ground is covered in time T . Show that the time taken for the entire fall to the ground is $5.45T$.
2. The Lady is 8 metres from the bus stop, when the Bus, starting from rest at the bus stop, starts to move off with an acceleration of 0.4 m s^{-2} . What is the least speed at which the Lady must run in order to catch the Bus?
3. A parachutist is descending at a constant speed of 10 feet per second. When she is at a height of 900 feet, her friend, directly below her, throws an apple up to her. What is the least speed at which he must throw the apple in order for it to reach her? How long does it take to reach her, what height is she at then, and what is the relative speed of parachutist and apple? Assume $g = 32 \text{ ft s}^{-2}$. Neglect air resistance for the apple (but not for the parachutist!)
4. A lunar explorer performs the following experiment on the Moon in order to determine the gravitational acceleration g there. He tosses a lunar rock upwards at an initial speed of 15 m s^{-1} . Eight seconds later he tosses another rock upwards at an initial speed of 10 m s^{-1} . He observes that the rocks collide 16.32 seconds after the launch of the first rock. Calculate g and also the height of the collision.
5. Mr A and Mr B are discussing the merits of their cars. Mr A can go from 0 to 50 mph in ten seconds, and Mr B can go from 0 to 60 mph in 20 seconds. Mr B gives Mr A a start of 12 seconds. How long does he take to catch Mr A, and how far have the cars travelled by then?

Answers I make the answers as follows. got any of them wrong. 4 Let me know (universe@uvvm. uvic. ca) if you think I have 2. 2.53 ms^{-1} 3. 230 ft s^{-1} , 7.5 s , 825 ft , 0 ft s^{-1} . 4. 1.64 ms^{-2} 26.4 m 5. 95 s , 1 mile 10 furlongs. 3. Motion in which the Resistance is Proportional to the Speed. If the only force on a

body is a resistive force that is proportional to its speed, the equation of motion is $m \frac{dv}{dt} = -bv$. 6.3.1 One thinks, for example, of Stokes's equation for the laminar motion of a sphere through a viscous fluid, in which the resistive force is $6\pi\eta r v$, where η is the coefficient of dynamic viscosity. If we divide both sides of the equation by the mass m , we obtain $\frac{dv}{dt} = -\gamma v$, where $\gamma = b/m$ is the damping constant. It has dimension T^{-1} and SI units s^{-1} . 6.3.2 3a. Resistive force only It is difficult to imagine a real situation in which the one and only force is a resistive force proportional to the speed. A body falling through the air won't do, because, in addition to the resistive force, there is the acceleration due to gravity. Perhaps we could imagine a puck sliding across the ice. The ice would have to be presumed to be completely frictionless, and the only force on the puck would be the resistance of the air. It is a slightly artificial situation, because we want the puck to be going so fast that the frictional force is negligible compared with the air resistance, but not so fast that the airflow is turbulent - but we need to start somewhere. The frictional force is, at least to a very good approximation, not a function of speed, but is constant, and we shall start by assuming that it is negligible and that the only horizontal force on the puck is air resistance and that the air resistance is proportional to the speed. In this case, the equation of motion is indeed equation 6.3.2. To obtain the first time integral, we write as v and the first time integral is readily found to be $V = v_0 e^{-\gamma t}$. 6.3.3 Here v_0 is the initial speed. This is illustrated in figure VI.2

FIGURE VI.2 time The speed is reduced to half of the initial speed in a time $\ln 2 / \gamma$. 6.3.4 The second time integral is found by writing v in equation 6.3.3 as dx/dt . Integration, with initial condition $x = 0$ when $t = 0$, gives $x = x_0 (1 - e^{-\gamma t})$. 6.3.5 where $x_0 = v_0 / \gamma$. This is illustrated in figure VI.3. It is seen that the puck travels an eventual distance of x_0 , but only after an infinite time.

FIGURE VI.3 time We can obtain the space integral either by eliminating t from between the two time integrals, or by writing the equation of motion as $v \frac{dv}{dx} = -\gamma v$. 6.3.6 dx With initial condition $v = v_0$ when $x = 0$, this becomes $v = v_0 e^{-\gamma x}$, 6.3.7 which is illustrated in figure VI.4. The speed drops linearly with distance (but exponentially with time) reaching zero after having travelled a finite distance $x = v_0 / \gamma$ in an infinite time.

FIGURE V1.4 Xoo x This analysis has assumed that the only force was the resistive force proportional to the speed. In the case of our imaginary ice puck, we were assuming that the resistive force was that of the air, the friction being negligible. Of course, as the puck slows down and the resistive force becomes less, there will come a point when the frictional force is negligible compared with the ever-decreasing air resistance, so that the above equations no longer accurately describe the motion. We shall come back to this point in subsection 3c. 3b. Body falling under gravity in a resisting medium, resistive force proportional to the speed. We are here probably considering a small sphere falling slowly through a viscous liquid, with laminar flow around the sphere, rather than a skydiver hurtling through the air. In the latter case, the airflow is likely to be highly turbulent and the resistance proportional to a higher power of the speed than the first. We'll use the symbol y for the distance fallen. That is to say, we measure y downwards from the starting point. The equation of motion is $\frac{dy}{dt} = g - kv$, 6.3.8 where g is the gravitational acceleration. The body reaches a constant speed when $\frac{dy}{dt}$ becomes zero. This occurs at a speed $v_5 = g/k$, which is called the terminal speed.

To obtain the first time integral, we write the equation of motion as 6.3.9 or $\frac{dy}{dt} = g - kv$. 6.3.10 CAUTION. At this point it might be tempting to write $\frac{dy}{dt} = g - kv$. 6.3.11 DON'T! In the middle of an exam, while covering this derivation that you know so well, you can suddenly find yourself in inextricable difficulties. The thing to note is this. If you look at the left hand side of the equation, you will anticipate that a logarithm will appear when you integrate it. Keep the denominator positive! Some mathematicians may know the meaning of the logarithm of a negative number, but most of us ordinary mortals do not - so keep the denominator positive! With initial condition $v = 0$ when $t = 0$, the first time integral becomes $v = \frac{g}{k}(1 - e^{-kt})$. 6.3.12 This is illustrated in figure VI.5. FIGURE VI.5 terminal Speed time Students will have seen equations similar to this before in other branches of physics - e.g. growth of charge in a capacitor or growth of current in an inductor. That is why learning physics becomes easier all the time, because you have seen it all before in quite different contexts.

9 Perhaps you have already noticed that third-year physics is easier than second-year physics; just think how much easier fourth-year is going to be!

At any rate, u approaches the terminal speed asymptotically, never quite reaching it, but reaching half of the terminal speed in time $(\ln 2)/\gamma = 0.693/\gamma$ (you have seen that before while studying radioactive decay), and reaching $(1 - e^{-1}) = 63\%$ of the terminal speed in time $1/\gamma$. If the body is thrown downwards, so that its initial speed is not zero but is $u = u_0$ when $t = 0$, you will write the equation of motion either as equation 6.3.10 or as equation 6.3.11, depending on whether the initial speed is slower than or faster than the terminal speed, thus ensuring that the denominator is kept firmly positive. In either case, the result is $u = v_t + (u_0 - v_t)e^{-\gamma t}$ 6.3.13 Figure VI.6 shows w as a function of t for initial conditions $w_0 = 0$, b , $2b$. T_w , FIGURE VI.6 time Returning to the initial condition $v = 0$ when $t = 0$, we readily find the second time integral to be

10 6.3.14 You should check whether this equation is what is expected for when $t = 0$ and when t approaches infinity. The second time integral is shown in figure VI.7. FIGURE VI.7 time The space integral is found either by eliminating t between the first and second time integrals, or by writing 3) as $v dv/dy$ in the equation of motion: 6.3.15 whence $y = -\frac{v_t^2}{g} \ln(1 - \frac{v}{v_t})$ 6.3.16 $\frac{v}{v_t} = 1 - e^{-\frac{g y}{v_t^2}}$

11 This is illustrated in figure VI.8. Notice that the equation gives y as a function of u , but only numerical calculation will give u for a given y . FIGURE VI.8 distance Problem: Assume $g = 9.8 \text{ m s}^{-2}$. A particle, starting from rest, is dropped through a medium such that the terminal speed is 9.8 m s^{-1} . How long will it take to fall through 9.8 m ? We are asked for t , given y , and we know the equation relating t and y - it is the second time integral, equation 6.3.13 - so what could be easier? We have $\frac{v}{v_t} = g t / v_t = 1 - e^{-g t / v_t}$, so equation 6.3.14 becomes $9.8 = 9.8 t - 9.80 e^{-t}$ and suddenly we find that it is not as easy as expected! The equation can be written $f(t) = t + e^{-2} = 0$. 6.3.17 6.3.18

12 For Newton-Raphson iteration we need $f'(t) = 1 - e^{-2}$ 6.3.19 and, after some rearrangement, the Newton-Raphson iteration ($t \rightarrow t - f/f'$) becomes $t \rightarrow t - \frac{t + e^{-2}}{1 - e^{-2}}$ 6.3.20 e^{-1} (It may be noticed that 6.3.20, which derives from the Newton-Raphson process, is merely a rearrangement of equation 6.3.18.) Starting with an exceedingly stupid first guess of $t = 100 \text{ s}$, the iterations proceed as follows: 100.000 000 000 2.000 000 000 1.843 482 357 1.841 406 066 1.841 405 661 1.841 405 660 s Problem: Assume $g = 9.8 \text{ m s}^{-2}$. A particle,

starting from rest, falls through a resisting medium, the damping constant being $\lambda = 1.96 \text{ s}^{-1}$ (i.e. $0 = 5 \text{ m s}^{-1}$). How fast is it moving after it has fallen 0.3 m? We are asked for v , given y . data, we obtain From this, We want the space integral, equation 6.3.16. $f(v) = 51\ln(1-0.2v) + v + 0.588 = 0$. $f'(v) = v/(v - 5) - 1$, The Newton-Raphson process ($t \rightarrow t_{f'}$), after some algebra, arrives at On substituting the 6.3.21 6.3.22 $-u(5 \ln(0.2u) + 0.588) u(5 \ln u - 7.459189560) v - +5 = +5$, 6.3.23 where $u = 5 - v$. This time Newton-Raphson does not allow us the luxury of an exceedingly stupid first guess, but we know that the answer must lie between 0 and 5 m s^{-1} , so our moderately intelligent first guess can be $v = 2.5 \text{ m s}^{-1}$

13 Newton-Raphson iterations: $v = 2.500 \ 000 \ 000 \ 2.122 \ 264 \ 100 \ 2.051 \ 880 \ 531 \ 2.049 \ 766 \ 247 \ -1 \ 2.049 \ 764 \ 400 \text{ m s}^{-1}$ Problems Here are four problems concerning a body falling from rest such that the resistance is proportional to the speed. Assume that $g = 9.8 \text{ m s}^{-2}$. Answers to questions 1 - 3 are to be given to a precision of 0.0001 seconds. 1. A particle falls from rest in a medium such that the damping constant is $\lambda = 1.0 \text{ s}^{-1}$. How long will it take to fall through 10 m? 2. It takes t seconds to fall through y metres. Construct a table showing t for 201 values of y going from 0 to 20 metres in steps of 0.1 metre. 3. Construct a table showing t for 201 values of y going from 0 to 20 metres in steps of 0.1 metres for $\lambda = 0.0, 0.5, 1.0, 1.5, 2.0 \text{ s}^{-1}$. The table is to have six columns. The first column gives the distance fallen to a precision of 0.1 metres. The remaining five columns will give the time, to a precision of 0.0001 seconds, that the body takes to fall a given distance, to a precision of 0.0001 seconds 4. Draw, by computer, a graph showing t (the dependent variable, plotted vertically) versus y (plotted horizontally) for the five values of λ in question 3. These four problems are in order of increasing difficulty. The first is merely an exercise in solving an implicit equation (equation 6.3.14) numerically, and might serve as an introductory example of how, for example, to solve an equation by Newton-Raphson iteration (I make the answer 1.8656 s.) The last two, if started from scratch, could well take up an entire afternoon before it is solved to one's complete satisfaction. It might be observed that the graphs of question 4 could be drawn fairly easily by calculating y explicitly as a function of t , thus obviating the necessity of Newton-Raphson iteration. No such short cuts, however, can be made for constructing the table of question 3. In fact I solved questions 3 and 4 in just a few minutes - but I did not start from scratch. As you progress through your

scientific career, you will become aware that there are certain operations that you encounter time and time again. To do questions 3 and 4, for example, you need to be

14 able to solve an equation by Newton-Raphson iteration; you need to be able to construct a table of a function $y = f(x; a)$, or in this case $t = f(y; 7)$; and you need to be able to instruct a computer to draw graphs of tabulated values. I learned long ago that all of these are problems that crop up frequently, and I therefore long ago wrote short programs (only a few lines of Fortran each) for doing each of them. All I had to do on this occasion was to marry these existing programs together, tailored to the particular functions needed. Likewise a student will recognize similar problems for which he or she frequently needs a solution. You should accumulate and keep a set of these small programs for use in the future whenever you may need them. For example, this is by no means the last time you will need Newton-Raphson iteration to solve an equation. Write a Newton-Raphson program now and keep it for future occasions! 3c. Body thrown vertically upwards, initial speed v_0 If we measure y upwards from the ground, the equation of motion is 3): $-g = dv/dt$. 6.3.24 The first time integral is $v = -gt + (v_0 + gt)e^{T/t}$ 6.3.25 and this is shown in figure VI.9. FIGURE VI.9 time It reaches a maximum height after time T , when $v = 0$ (at which time the acceleration is just $-g$):

15 $T = -v_0/g$ In 1+ . 6.3.26 The second time integral (obtained by writing v as dy/dt in equation 6.3.25) and the space integral (obtained by writing y as $v dv/dy$ in the equation of motion) require some patience, but the results are $Y = (v_0^2 + 2gt^2)/2g$ 6.3.27 $v = v_0 - gt$ 6.3.28 These are illustrated in figures VI. 10 and VI. 11. FIGURE VI.10 H time T

16 FIGURE VI.11 distance It might be remarked that you cannot, on this occasion, obtain the space integral by algebraic elimination of t between the two time integrals. You should verify that, for small t , $2u \approx gt$ and $y \approx P_0 t$; and for small y , $p^2 \approx P_0^2 - 2gy$. The maximum height H is reached when $v = 0$ in equation 6.3.28: 6.3.29 For a puck sliding along ice and subject to air resistance and a frictional force, the equation of motion is $= -pg - kv^2$. 6.3.30 This is very similar to equation 6.3.24, with the substitution of pg for g , and all the subsequent equations and conclusions follow, including the maximum distance travelled and the time taken to get there - except that the equations

seem to predict that, when the speed becomes zero, the

17 acceleration is $-g$, and the puck will subsequently start to move backwards (just as the ball thrown vertically upwards will start to fall). The reader is invited to cogitate upon this and determine just where there is a difference between the motion of the one and the motion of the other. More problems on motion in which the resistance is proportional to the speed 5. A particle is projected upwards. one third of the speed of projection. projection. The average speed between launch and the highest point is Calculate the ratio of the terminal speed to the speed of 6. A particle is projected upwards. Show that the average speed between launch and the highest point cannot be more than half the speed of projection. 7. A particle is projected upwards in a medium of damping constant 1.2 s^{-1} . It takes twice as long to come down as to go up. What was the initial speed? ($g = 9.8 \text{ m s}^{-2}$.) A particle falls from rest. The damping constant is g and the gravitational acceleration is g . Expand y as a power series in t up to t^3 . Expand y as a power series in v up to v^3 . Two identical particles are moving in a medium in which their terminal speed is rS . One is launched vertically upwards from the ground with initial speed U . At the same instant, the other is launched vertically downwards, also with initial speed U , from a point at a vertical height h above the point of projection of the first particle. Show that, provided that $2U^2 > gh$, the two particles meet after a time $\frac{h}{U}$. A particle is launched vertically upwards with a speed U times the terminal speed. Show that on return it hits the ground with a speed U times the terminal speed, where $\ln(1 + U/r) = U/r$. Draw a graph of V (vertical axis) versus U (horizontal axis) for U going from 0 to 2. Answers. 5. $t/v_0 = 0.1320$. 7. 29.96 m s^{-1} ; $Y = + \frac{1}{2}v_0^2 - \frac{1}{2}g$

18 4. Motion in which the Resistance is Proportional to the Square of the Speed. There are not really any new principles; it is just a matter of practice with slightly more difficult integrals. We assume that the resistive force per unit mass is $2k$. Here, although k plays a somewhat similar role to the $\frac{1}{2}g$ of section 3, it is not exactly the same thing as g , and indeed it is not dimensionally the same as g . What are the dimensions, and the SI units, of k ? 4a. Resistive force only We'll imagine a puck sliding along a frictionless surface against turbulent air resistance. The equation of motion is: $\frac{dv}{dt} = -kv^2$. 6.4.1 By this time we assume that the student knows how to obtain the first

and second time integrals and the space integral. The actual integrations may be slightly more difficult, but we leave it to the reader to obtain the results v - 6.4.2 $1 + kv_0 t \ln(1 + kv_0 t)$ x - , 6.4.3 $k v = V_0 e$ 6.4.4 These are illustrated in figures VI. 12,13,14. Note that, provided that equation 6.4.1 accurately describes the entire motion (which may not be the case in a practical situation), there is no finite limit to x, nor does the speed drop to zero in any finite time.

FIGURE V1.12 time F1GURE V1.13 time

FIGURE V1.14 distance 4b. Body falling under gravity in a resisting medium, resistive force proportional to the square of the speed The equation of motion is $f_i = g - kv^2$. 6.4.4 The limiting, or terminal, speed, which is reached when the acceleration is zero, is given by $15 = J'/k$, 6.4.5 so that the equation of motion can be written $\dot{v} = k(15^2 - v^2)$ 6.4.6 The first and second time integrals and the space integral can be found in the usual way, though with perhaps a little more effort: $v = 15 \tanh(k15t)$, 6.4.7

$21 \ln[\cosh(kDt)]$, $y = 6.4.8 \frac{1}{2} = 20 \frac{1}{2} e^{2ky}$ 6.4.9 These are illustrated in figures VI. 15, 16,17. The inverse relations corresponding to equations 6.4.7 and 6.4.8 can be found from the usual formulas for the inverse hyperbolic functions $\tanh^{-1} x = \ln \frac{1+x}{1-x}$ and $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$ --X FIGURE VI15 time FIGURE V1.16 time

22 F1GURE V1.17 distance 4C. of the speed. The equation of motion is $\ddot{x} = -g(1 + \frac{v^2}{c^2})$, where $t_5 = x_0/k$. The first time integral is $v = b \tan(cz - kvt)$, where $\tan c = v_0/rS$. The time T to maximum height is found by putting $v = 0$: Body thrown vertically upwards, initial speed u_0 , resistance proportional to the square The second time integral is 6.4.10 6.4.11 $\tan^{-1} (V_0/t_3) T = 6.4.12 kb \ln \cos(C - kbt) - Y: k L \cos$ Although this expression "looks" periodic, it is valid only for $t < T = \pi/(kO)$. 6.4.13

The space integral is 23 $u^2 = \frac{1}{2} u_0^2 + t^2 e^{-2t^2}$ 6.4.14 The maximum height H is reached when the speed is zero, whence $H = k \ln(1 + u_0^2)$. 6.4.15 The first and second time integrals (equations 6.4.11 and 13) and the space integral (equation 6.4.14) are illustrated in figure VI.18, 19 and 20. FIGURE V1.18 time

24 FIGURE V1.19 H time FIGURE V1.20 distance H Problem A particle is projected vertically upward with initial speed equal to $\tan c$ times the terminal speed, the resistance being proportional to the square of the speed. Show that on return the particle hits the ground with speed $\sin c$ times the terminal speed.

25

1 CHAPTER 7 PROJECTILES 1. No Air Resistance We suppose that a particle is projected from a point O at the origin of a coordinate system, the y-axis being vertical and the x-axis directed along the ground. The particle is projected in the xy-plane, with initial speed U_0 at an angle c to the horizon. At any subsequent time in its motion its speed is U and the angle that its motion makes with the horizontal is θ . The initial horizontal component of the velocity is $U_0 \cos c$, and, in the absence of air resistance, this horizontal component remains constant throughout the motion. I shall also refer to this constant horizontal component of the velocity as u . I.e. $u = U_0 \cos c =$ constant throughout the motion. The initial vertical component of the velocity is $U_0 \sin c$, but the vertical component of the motion is decelerated at a constant rate g . At a later time during the motion, the vertical component of the velocity is $U \sin \theta$, which I shall also refer to as v . In the following, I write in the left hand column the horizontal component of the equation of motion and the first and second time integrals; in the right hand column I do the same for the vertical component.

Horizontal. Vertical = 0 $\frac{d^2x}{dt^2} = 0$ $\frac{d^2y}{dt^2} = -g$ 7.1.1a,b $\frac{dx}{dt} = u = U_0 \cos c$ $\frac{dy}{dt} = v = U_0 \sin c - gt$ 7.1.2a,b $x = U_0 t \cos c$ $y = U_0 t \sin c - \frac{1}{2}gt^2$ 7.1.3a,b

The two equations 7.1.3a,b are the parametric equations to the trajectory. In vector form, these two equations could be written as a single vector equation: $\mathbf{r} = U_0 \mathbf{e}_c t - \frac{1}{2}g t^2$ 7.1.4 Note the + sign on the right hand side of equation 7.1.4. The vector g is directed downwards. The xy-equation to the trajectory is found by eliminating t between equations 7.1.3a and 7.1.3b to yield: gx^2 7.1.5 $y = x \tan c - \frac{g}{2U_0^2 \cos^2 c} x^2$

Now, re-write this in the form $x^2 + (-2u \tan c)x = -2u^2 \tan^2 c$ (Add to each side (half the coefficient of x)² in order to "complete the square" on the left hand side, and, after some algebra, it will be found that the equation to the trajectory can be written as: $(x-A)^2 = -4a(y-B)$ 7.1.6 where $A = \frac{u^2}{g \sin c \cos c}$ $B = \frac{u^2 \sin^2 c}{2g}$ 7.1.7 $a = \frac{u^2 \cos^2 c}{2g}$ 7.1.8 $\tan c = \frac{u \sin c}{u \cos c}$ 7.1.9 Having re-arranged equation 7.1.5

in the form 7.1.6, we see that the trajectory is a parabola whose vertex is at (A, B). The range on the horizontal plane is $2A$, or $V_0^2 \sin^2 c / g$. The greatest range on the horizontal plane is obtained when $\sin 2c = 1$, or $c = 45^\circ$. The greatest range on the horizontal plane is therefore V_0^2 / g . The maximum height reached is B , or $V_0^2 \sin^2 c / (2g)$. The distance between vertex and focus is a , or $V_0^2 \cos^2 c / (2g)$. The focus is above ground if this is less than the maximum height, and below ground if it is greater than the maximum height. That is, the focus is above ground if $\cos^2 c < \sin^2 c$. That is to say, the focus is above ground if $c > 45^\circ$ and below ground if $c < 45^\circ$. The radius of curvature p anywhere along the trajectory can be found using the usual formula $(1 + y'^2)^{3/2} / y''$. At the top of the trajectory, $y' = 0$, so that $p = 1/y''$. Alternatively (in case one y'' has forgotten or is unfamiliar with the "usual formula"), we note that the speed at the top of the path is just equal to the (constant) horizontal component of the velocity $V_0 \cos c$. We can then equate the centripetal acceleration ($V_0^2 \cos^2 c / p$) to g and hence obtain: $V_0^2 \cos^2 c = pg$ 7.1.10

3 By subtracting this from our expression for the maximum height of the projectile, we find that the height of the centre of curvature above the ground is $V_0^2 (1 - 3 \cos^2 c) / (2g)$. The centre of curvature is above ground if $c > 54.74^\circ$. The range r on a plane inclined at an angle θ to the horizontal can be found by substituting $x = r \cos \theta$ and $y = r \sin \theta$ in the equation 7.1.5 to the trajectory. This results, after some algebra, in $r = V_0^2 [\sin(2c - \theta) - \sin \theta] / (g \cos \theta)$. This is greatest when $2c - \theta = 90^\circ$; i.e. when the angle of projection bisects the angle between the inclined plane and the vertical. The maximum range is $r = V_0^2 (1 + \sin \theta) / g$ 7.1.12 This is the equation, in polar coordinates, of a parabola, and this parabola, when rotated about its vertical axis, describes a paraboloid, known as the paraboloid of safety. It is the envelope of all possible trajectories with an initial speed V_0 . If a gun is firing shells with initial speed V_0 , or a lawn sprinkler is ejecting water at initial speed V_0 , you are safe as long as you are outside the paraboloid of safety. Figure VII. 1 shows trajectories for $c = 20, 40, 60, 80, 100, 120, 140$ and 160 degrees, and, as a dashed line, the paraboloid of safety. Notice how the range changes with c and that it is greatest for $c = 45^\circ$. FIGURE VI

Problem. A gun projects a shell, in the absence of air resistance, at a initial angle θ to the horizontal. The speed of projection varies with angle of

projection and is given by Initial speed = $V_0 \cos \alpha$. Show that, in order to achieve the greatest range on the horizontal plane, the shell should be projected at an angle to the horizontal whose cosine c is given by the solution of the equation $3c^3 + 2c - 2\epsilon = 0$. Find the optimum angle to a precision of one arcminute. 2. Air resistance proportional to the speed. As in the previous section, I shall write the x-component of the equation of motion, and of the first and second time integrals, in the left hand column, and the y-component in the right-hand column. The x-component of the air resistance per unit mass is $-\epsilon u$ and the y-component is $-\epsilon v$. Here ϵ is the damping constant, defined in Chapter 6, section 3. The x- and y-components of the initial velocity are, respectively, $V_0 \cos \alpha$ and $V_0 \sin \alpha$. It should be readily seen that the equations of motion and their time integrals are as follows:

Horizontal	Vertical
$\ddot{x} = -\epsilon \dot{x}$	$\ddot{y} = -g - \epsilon \dot{y}$
$\dot{x} = V_0 \cos \alpha e^{-\epsilon t}$	$\dot{y} = u = V_0 \cos \alpha e^{-\epsilon t}$
$x = V_0 \sin \alpha (1 - e^{-\epsilon t}) / \epsilon$	$\dot{y} = g / \epsilon (1 - e^{-\epsilon t}) + V_0 \sin \alpha e^{-\epsilon t}$
$y = \frac{g}{2\epsilon^2} (1 - e^{-2\epsilon t}) - \frac{V_0 \sin \alpha}{\epsilon} (1 - e^{-\epsilon t})$	

(In case it is not "readily seen", for the horizontal motion refer to Chapter 6, section 3, especially equations 6.3.2, 6.3.3 and 6.3.5, and for the vertical motion refer to Chapter 6, section 3b, especially equations 6.3.24, 6.3.25 and 6.3.27.) It will be seen that, as $t \rightarrow 0$, $u \rightarrow 0$, $v \rightarrow -t$, $x \rightarrow 0$. The xy-equation to the trajectory is the t-eliminant of equations 6.2.3a and 6.2.3b. After a small amount of algebra this is found to be:

$y = \frac{g}{2\epsilon^2} (1 - e^{-2\epsilon x / V_0 \sin \alpha}) - \frac{V_0 \sin \alpha}{\epsilon} (1 - e^{-\epsilon x / V_0 \sin \alpha})$ This is illustrated in figure VII.2. 7.2.4 1.5 0:5 'FIGURE VII,2 1 Z .3 '4' 5 ϵ x.:(m). 7 The range on a horizontal plane is found by setting $y = 0$, to obtain either $x = -A \ln(1 - x/A)$ $A = V_0 \cos \alpha / \epsilon (V_0 \sin \alpha + g)$ where 7.2.5 7.2.6

Example. Then and Suppose $V_0 = 20 \text{ m s}^{-1}$, $\epsilon = 50 \text{ s}^{-1}$, $g = 9.8 \text{ ms}^{-2}$, $\epsilon = 1.96 \text{ s}^{-1}$, $A = (g/\epsilon^2) = 1.61387065 \text{ m}$, 6.55905724 m . Try to find the range on the horizontal plane, using either equation 7.2.5 or 7.2.6, to nine significant figures. Which equation works best? Newton-Raphson may fail with a stupid first guess - but it should not be difficult to make a fairly intelligent first guess. I should not tell you, but figure VII.2 was calculated using the data of this example. 3. Air resistance proportional to the square of the speed. Notation: V is the velocity, I is the speed. The horizontal and vertical components of the velocity are, respectively, $u = I \cos \theta$ and $v = I \sin \theta$. Here θ is the angle that the instantaneous velocity makes with the horizontal.

The resistive force per unit mass is kl^2 . The horizontal and vertical components of the resistive force are $kl^2 \cos \theta$ and $kl^2 \sin \theta$ respectively. The launch speed is V_0 and the launch angle (i.e. the initial value of θ) is α . Distance travelled from the launch point, measured along the trajectory, is s , and speed V . The equations of motion are: Horizontal: $\ddot{s} = -kl^2 \cos \theta$ 7.3.1 Vertical: $\ddot{s} = -g - kl^2 \sin \theta$ 7.3.2 These cannot be integrated as conveniently as in the previous cases, but we can get a simple relation between the horizontal component u of the speed and the intrinsic coordinate s . Thus, when we make use of $\dot{s} = V$, $V \cos \theta = u$, equation 7.3.1 takes the form $\ddot{s} = -ku$ 7.3.3 Integration, with initial condition $u = V_0 \cos \alpha$, yields $u = V_0 \cos \alpha [1 - ks]$ 7.3.4

7 We can also obtain an exact explicit intrinsic equation to the trajectory by consideration of the normal equation of motion. The intrinsic equation to any curve is a relation between the intrinsic coordinates (s, t) . The rate at which the slope angle changes as you move along the curve, i.e. $d\theta/ds$, is called the curvature at a point along the curve. If the slope is increasing with s , the curvature is positive. The reciprocal of the curvature at a point, $ds/d\theta$, is the radius of curvature at the point, denoted here by ρ . The normal equation of motion is the equation $F = ma$ applied in a direction normal to the curve. The acceleration appropriate here is the centripetal acceleration V^2/ρ or $V^2 d\theta/ds$. In a direction normal to the motion, the air resistance has no component, and gravity has a component $-g \cos \theta$. (It is minus because the curvature is clearly negative.) The normal equation of motion is therefore $v \frac{d\theta}{ds} = -g \cos \theta$ 7.3.5 But $V = u = V_0 \cos \alpha [1 - ks]$ 7.3.6 $\cos \theta = \frac{u}{V}$ Therefore $V_0 \cos \alpha : \cos \theta = \frac{ds}{dt} : g \cos \theta$ 7.3.7 ds Separate the variables, and integrate, with appropriate initial conditions: $\int \frac{ds}{V_0 \cos \alpha [1 - ks]} = \int \frac{dt}{g \cos \theta}$ From here it is good integration practice to show that the intrinsic equation is $\sec \theta \tan \theta - \sec \alpha \tan \alpha + \ln(\sec \theta + \tan \theta) - g \frac{O - e^{2kt}}{k} = \sec \alpha \tan \alpha + \ln(\sec \alpha + \tan \alpha) + \frac{V_0^2 \cos^2 \alpha}{2k}$ 7.3.8 7.3.9 Problem. If I asked you to program a computer to draw the path of a projectile in the absence of air resistance, there would be no difficulty - you would just use equation 7.1.5. And if I asked you to do the same in the case in which the resistance is proportional to the speed, you would just use equation 7.2.4. But instead, I am going to ask you to draw the path in the case in which the resistance is proportional to the square of the speed. We do not have a simple equation of the form $y = f(x)$; we have instead the intrinsic equation 7.3.9, and this is not so easy. I have not done it, but it ought to be possible.. For example, if we assume $\alpha = 45^\circ$, that $k = 1 \text{ m}^{-1}$, and that

the initial speed is such that $g/(kV_0) = 1$, equation 7.3.9 becomes: I think, if I have done my arithmetic correctly, that $e^{-s} = 2.147793575 - 3[\sec^2 t + \ln(\sec^2 t + \tan^2 t)]$, 7.3.10 and this is the graph that has to be drawn. The graph that has to be drawn, of course, is not s versus t - that's easy. You have to plot y versus x - that's more difficult! Have a go at it, and, if you manage to do it and to draw the path, accurately by computer, let me know (universe.uvm.uvic.ca), and how you did it, and we'll see if we can incorporate your graph - with due acknowledgments, of course - in these notes. I don't think there is a finite x , but see if you can find the range on the horizontal plane, and let me know if you can.

1 CHAPTER 8 IMPULSIVE FORCES 1. Introduction. As it goes about its business, a particle may experience many different sorts of forces. In Chapter 7, we looked at the effect of forces that depend only on the speed of the particle. In a later chapter we shall look at forces that depend only on the position of the particle. (Such forces will be called conservative forces.) In this chapter we shall look at the effect of forces that vary with time. Of course, it is quite possible that an unfortunate particle may be buffeted by forces that depend on its speed, on its position, and on the time - but, as far as this chapter is concerned, we shall be looking at forces that depend only on the time. Everyone knows that Newton's second law of motion states that when a force acts on a body, the momentum of the body changes, and the rate of change of momentum is equal to the applied force. That is, $F = dp/dt$. If a force that varies with time, $F(t)$, acts on a body for a time T , the integral of the force over the time, $\int F(t)dt$, is called the impulse of the force, and it results in a change of momentum Δp which is equal to the impulse. I shall use the symbol J to represent impulse, or the time integral of a force. Its SI units would be $N \cdot s$, and its dimensions MLT^{-1} , which is the same as the dimensions of momentum. Thus, Newton's second law of motion is $F = dp/dt$. 8.1.1 When integrated over time, this becomes $\int F dt = \Delta p$. 8.1.2 Likewise, in rotational motion, the angular momentum L of a body changes when a torque τ acts on it, such that the rate of change of angular momentum is equal to the applied torque: $\tau = dL/dt$. 8.1.3 If the torque, which may vary with time, acts over a time T , the integral of the torque over the time, $\int \tau dt$, is the angular impulse, which I shall denote by the symbol K , and it results in a change of the angular momentum: $K = \Delta L$. 8.1.4 The SI units of angular impulse are $N \cdot m \cdot s$, and the dimensions are ML^2T^{-1} , which are the same as those of angular momentum.

2 For example, suppose that a golf ball is struck by a force that varies with time as in figure VIII.1. I have drawn it for $t_0 = 1$, $t_c = 3$, and you will see that it is a reasonably plausible function. The club is in contact with the ball from time $t = -t_c$ to $t = t_c$. 0.6 0.4 0.2

FIGURE VIII.1 $F(t)$ vs t . If the ball, of mass m , starts from rest, what will be its speed v_z immediately after it leaves the club? The answer is that its new momentum, mv_z , will equal the impulse (or the time integral) of the above force: $\int_{-t_c}^{t_c} F(t) dt = mv_z$. 8.1.6 This is very easy to understand; if there is any difficulty it might be in the mechanics of working out this integral. It is good integration practice, but, if you can't do it after a reasonable effort, and you want a hint, ask me (universeuvm.uvic.ca) and I'll see what I can do. You should get

$v_z = 3c/c = 0.589/c$. 8.1.7 16 Example. Here is a very similar example, except that the integration is rather easier. Initially at rest, a ball is struck with a force that varies with time as in figure VIII.1. (It doesn't look quite like figure VIII.1.) impact? (I make it 37.7 m/s .) A ball of mass 500 g , $t_0 = 10 \text{ ms}$, $t_c = 3 \text{ ms}$. Draw (accurately, by computer) a graph of F versus t . How fast is the ball moving immediately after impact? 2. Problems In this section I offer a set of miscellaneous problems. In a typical problem it is assumed that an impulsive force or torque acts for only a very short time. By "a very short" time, I mean that the time during which the force or torque acts is very small or is negligible compared with other times that might be involved in the problem. For example, if a golf club hits a ball, the club is in contact with the ball for a time that is negligible compared with the time in which the ball is in the air. Or if a pendulum is subjected to an impulsive torque, the time during which the impulsive torque is applied is negligible compared with the period of the pendulum. In many problems, you will be told that a body is subjected to an impulse J . The easiest way to interpret this is to say that the linear momentum of the body suddenly changes by J . Or you may be told that the body is subjected to an impulsive torque K . The easiest way to interpret this is to say that the angular momentum of the body suddenly changes by K . In some of the problems, for example the first one, the body concerned is freely hinged about a fixed point; that is, it can freely rotate about that point. Before giving the first problem, here is a little story. One of the most inspiring lectures I remember going to was one given by a science educator. She complained that a professor, instead of inspiring his students with a love and appreciation of

the great and profound ideas of science and civilization, "infantalized" the class with a tiresome insistence that the class use blue pencils for velocity vectors, green for acceleration, and red for forces. I recognized immediately that this was a great way of imparting to students an appreciation of the profound ideas of physics, and I insisted on it with my own students ever since. In some of the following drawings I have used this colour convention, though I don't know whether your computer will reproduce the colours that I have used. In any case, I strongly recommend that you use the colour

4 convention so deprecated by the educator if you civilization, such as the ideas of impulsive forces. want to understand the great ideas of Problem 2.1 P hto  FIGURE VIII.2 $OC = h$ $OA = x$ In figure VIII.2, a body is free to rotate about a fixed axis O. The centre of mass of the body is at C. The distance OC is h. The body is struck with a force of impulse J at A, such that $OA = x$. The mass of the body is m. Its rotational inertia about C is mk^2 , and its rotational inertia about O is $m(k^2 + h^2)$. As a result of the blow, the body will rotate with angular speed ω and the centre of mass will move forward with linear speed $h\omega$. One of the questions in this problem is to calculate ω . The body will also push with an impulsive force against the axis at O. It is not immediately obvious whether the body will push upwards against the axis in the same direction as J, or whether the left hand end of the body will swing downwards and the body will push downwards on the axis. You will probably agree that if A is very close to O, the body will push upwards on the axis, but if A is near the right hand end, the body will push downward on the axis. In the figure, I am assuming that the body pushes upwards on the axis; the axis therefore pushes downwards on the body, with a force of impulse P, and what the figure shows is the two impulsive forces that act on the body. The second question to be asked in this problem is to find P in terms of J and x. If we are right in our intuitive feeling that P acts upwards or downwards according to the position of A - i.e. on where the body is struck - there is presumably some position of A such that the reactive impulse of the axis on the body is zero. Indeed there is, and the position of A that gives rise to zero reactive impulse at A is called the centre of percussion, and a third question in this problem is to find the position of the centre of percussion. Where on the bat should you hit the baseball if you want zero impulsive reaction on your wrists? Where should you position a doorstop so as to result in zero reaction on the door hinges? Never let it be said that theoretical physics does

not have important practical applications. The very positioning of a door-stop depends on a thorough understanding of the principles of classical mechanics. The net upward impulse is $J - P$, and this results in a change in linear momentum $m\dot{y}$: $J - P = m\dot{y}$. 8.2.1

The impulsive torque about O is Jx , and this results in a change in angular momentum $I\dot{\theta}$; that is to say $m(k^2 + h^2)\dot{\theta} = Jx$. 8.2.3

These two equations enable us to solve for the two unknowns $\dot{\theta}$ and P .

Indeed, equation 8.2.3 gives us to immediately, and elimination of $\dot{\theta}$ between the two equations gives us P : $P = J \frac{k^2 + h^2}{x}$. 8.2.4 If the right hand side of equation 8.2.4 is positive, then figure VIII.2 is correct: the axis pushes down on the body, and the body pushes upwards on the axis. That is, P acts downwards if $k^2 + h^2 < x^2$, and upwards if $x > \sqrt{k^2 + h^2}$. The position of the centre of percussion is $x = \sqrt{k^2 + h^2}$. If the body is a uniform rod of length l , O is at one end of the rod, then $k^2 = \frac{1}{12}l^2$ and $h = l$, so that, in this case, $x = \frac{1}{2}l$. This is where you should position a door-stop. It is also where you should hit a baseball with the bat - if the bat is a uniform rod. However, I admit to not knowing a great deal about baseball bats, and if such a bat is not a uniform rod, but is, for example, thicker and heavier at the distal than the proximal end, the centre of percussion will be further towards the far end. Problem 2.2 FIGURE VIII.3 A heavy rod, of mass m and length $2l$, hangs freely from one end. It is given an impulse J as shown at a point at a distance x from the upper end. Calculate the maximum angular height through which the rod rises.

6 We can use equation 8.2.3 to find the angular speed $\dot{\theta}$ immediately after impact. In this equation, $m(k^2 + h^2)$ is the rotational inertia of the rod about its end, which is $\frac{4}{3}ml^2$, so that $\dot{\theta} = \frac{3Jx}{4ml^2}$. The kinetic energy immediately after impact is $\frac{1}{2}I\dot{\theta}^2 = \frac{9J^2x^2}{8ml^2}$, and we have to equate this to the subsequent gain in potential energy $mg(1 - \cos \theta)$. Thus $\cos \theta = 1 - \frac{9J^2x^2}{8gm^2l^2}$. To get the rod to swing through 180° , the angular impulse applied must be $J = \frac{2}{3}mgl$. Problem 2.3 FIGURE VIII.4 A uniform rod of mass m and length $2l$ is freely hinged at one end O. A mass cm (where c is a constant) is attached to the rod at a distance x from O. An impulse J is applied to the other end of the rod from O. Where should the mass cm be positioned if the linear speed of the mass cm immediately after the application of the impulse is to be greatest? The angular impulse about O is

21J. The rotational inertia about O is $\frac{1}{2}ml + cmx$. If ω is the angular speed immediately after the blow, the angular momentum is $(\frac{1}{2}ml + cmx)\omega$. If we equate this to the impulse, we find $6U\omega = m(4l^2 + 3cx^2)$. The linear speed of the mass cm is x times this, or $2l$ when $x = x/c = \frac{4}{3}l$. By calculus, this is greatest

Problem 2.4 A Y FIGURE VIII.5 p x J A uniform rod is of mass m and length $2l$. An impulse d is applied as shown at a distance x from the mid-point of the rod. P is a point at a distance y from the mid-point of the rod. Does P move forward or backward? Which way does A move? The first thing we can do is to find the linear speed u of the centre of mass of the rod and the angular speed ω of the rod. We do this by equating the impulse to the increase in linear momentum and the moment of the impulse (i.e. the angular impulse) to the increase in angular momentum: and Thus we know both u and ω . J dx: $\frac{1}{2}ml\omega = d \cdot 3x$ 12 The forward velocity of P is $u + y\omega$. That is to say -- q- --. This is positive if $y > \frac{2}{3}x$ but negative otherwise. For the point A, $y = -l$, so that A will move forward if $x < l/3$, and it will move backwards otherwise.

Problem 2.5 B FIGURE VIII.6 A spherical planet, mass m , radius a , is struck by an asteroid with an impulse d as shown, the impact parameter being x . P is a point on the diameter, a distance y from the centre of the planet. Does P move forward or backward? Which way does A move? As in the previous problem, we can easily find u and ω : $d = mu$ and $dx = ma^2\omega$. ($5my$) The forward velocity of P is $u + y\omega$. That is to say $1 + 2a/y$. This is positive if $y > -5x$ but negative otherwise. For the point A, $y = 4a$, so that A will move forward if $x < 2a/5$, and it will move backward otherwise. That is to say A will move backwards if the blow is more than 70% of the way from A to B.

Problem 2.6 A h FIGURE VIII.7 A hoop, radius a , mass m , moving at speed w , hits a kerb of height h as shown. Will it mount the kerb, or will it fall back? The only impulse acts at the point A. The moment of the impulse about A is therefore zero, and therefore there is no change in angular momentum with respect to the point A. Before impact, the angular momentum with respect to the point A is $mva + my(a-h) = mv(2a-h)$. Let ω be the angular speed about A after impact. The angular momentum about A is then $2ma^2\omega$. These are equal, so that $(2a - h)\omega = 2a^2\omega$. If it is to mount the kerb, the new

kinetic energy ($\frac{1}{2}mv^2$) must be greater than the potential energy that is to be overcome, mgh . Therefore $v > \sqrt{2ah}$

Problem 2.7 Three equal particles, A, B and C, each of mass m , are joined by light inextensible strings as shown, the angle BAC being 60° . A is given a sharp tug of impulse J as shown. Find the initial velocities of the particles and the initial tensions in the strings. Let the initial tensions in BA and AC be T and T' respectively. Let the initial velocity of A be $u_i + v_j$. Then the initial velocity of B is u_i and the initial velocity of C is $-u_i - v_j$ towards A. The equations of impulsive motion are: For B: $mu = T$ For C: $m(-u_i - v_j) = T'$ For A: $mu = J - T - T'$ The solutions of these equations are:

Problem 2.8 Two rods, each of mass m and length $2l$, are freely jointed as shown. One of them is given an impulse J as shown. What happens then is that the end of one rod gives the end of the other an impulsive kick P , and the other gives the one an equal kick in the opposite direction. The centre of mass of the system moves forward with speed u and the two rods rotate with angular speeds ω and 2ω . The problem is to determine P , u , ω and 2ω . The equations of impulsive motion are: LH rod, translation: $J - P = m(u + l\omega)$, RH rod, translation: $P = m(u - l2\omega)$ LH rod, rotation: $Pl = -\frac{1}{2}m\omega l^2$ RH rod, rotation: $Pl = \frac{1}{2}m(2\omega)^2 l^2$ The solutions to these equations are: $J = 3P$, $\omega = \frac{J}{8ml}$, $P = J/8$.

Problem 2.9 Two rods, each of mass m and length $2l$, are freely jointed initially at right angles. They are dropped on to a smooth horizontal table and strike it with speed V . Find the rate $\dot{\theta}$ at which the rods splay apart immediately after impact. We consider the dynamics of the right hand rod. On impact, it experiences a vertical impulse J at its lower end, and it experiences a horizontal impulse P (from the other rod) at its upper end. Immediately after impact, let the components of the velocity of the centre of mass of the right hand rod be u and v , and the angular speed of the rod is the required quantity. From geometry, we have $x = l \sin \theta$ and $y = l \cos \theta$ and hence $\dot{x} = l \cos \theta \dot{\theta}$ and $\dot{y} = -l \sin \theta \dot{\theta}$. The impulsive equations of motion are $P = mu$

and $d = m(V - v)$. After that, some algebra results in $\dot{\theta} = \frac{J}{2ml}$ **Problem 2.10** A square plate is spinning about a vertical diameter at

angular speed ω . It strikes a fixed obstacle at the corner A, so that it subsequently spins about a vertical axis through A at angular speed ω . Find ω . Since the impulse is at A, the moment of the impulse about A is zero, so that angular momentum about A is conserved. The relevant moments of inertia can be calculated from the information in Chapter 2, and hence we obtain $\omega = \omega_0$.

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CHAPTER 9 CONSERVATIVE FORCES Introduction. In Chapter 7 we dealt with forces on a particle that depend on the speed of the particle. In Chapter 8 we dealt with forces that depend on the time. In this chapter, we deal with forces that depend only on the position of a particle. Such forces are called conservative forces; While only conservative forces act, the sum of potential and kinetic energies is conserved. Conservative forces have a number of properties. One is that the work done by a conservative force (or, what amounts to the same thing, the line integral of a conservative force) as it moves from one point to another is route-independent. The work done depends only on the coordinates of the beginning and end points, and not on the path taken to get from one to the other. It follows from this that the work done by a conservative force, or its line integral, round a closed path is zero. (If you are reminded here of the properties of a function of state in thermodynamics, all to the good.) Another property of a conservative force is that it can be derived from a potential energy function. Thus for any conservative force, there exists a scalar function $V(x, y, z)$ such that the force is equal to $-\text{grad } V$, or $-\nabla V$. In a one-dimensional situation, a sufficient condition for a force to be conservative is that it is a function of its position alone. In two- and three-dimensional situations, this is a necessary condition, but it is not a sufficient one. That a conservative force must be derivable from the gradient of a potential energy function and that its line integral around a closed path must be zero implies that the curl of a conservative force must be zero, and indeed a zero curl is a necessary and a sufficient condition for a force to be conservative. This is all very well, but suppose you are stuck in the middle of an exam and your mind goes blank and you can't think what a line integral or a grad or a curl are, or you never did understand them in the first place, how can you tell if a force is conservative or not? Here is a rule of thumb that will almost never fail you: If the force is the tension in a stretched

elastic string or spring, or the thrust in a compressed spring, or if the force is gravity or if it is an electrostatic force, the force is conservative. If it is not one of these, it is not conservative. Example. A man lifts up a basket of groceries from a table. Is the force that he exerts a conservative force?

Answer: No, it is not. The force is not the tension in a string or a spring, nor is it electrostatic. And, although he may be fighting against gravity, the force that he exerts with his muscles is not a gravitational force. Therefore it is not a conservative force. You see, he may be accelerating as he moves the basket up, in which case the force that he is exerting is greater than the weight of the groceries. If he is moving at constant speed, the force he exerts is equal to the weight of the groceries. Thus the force he exerts depends on whether he is accelerating or not; the force does not depend only on the position. Example. But you are not in an exam now, and you have ample time to remind yourself what a curl is. Each of the following two forces are functions of position only - a necessary condition for them to be conservative. But it is not a sufficient condition. In fact one of them is

conservative and the other isn't. You will have to find out by evaluating the curl of each. The one that has zero curl is the conservative one. When you have identified it, work out the potential energy function from which it can be derived. in other words, find $V(x,y,z)$ such that $F = -\nabla V$. i. $F = (3x^2z - 3y^2z)\mathbf{i} - 6xyz\mathbf{j} + (x^3 - xy^2)\mathbf{k}$ ii. $F = axyz\mathbf{i} + oxyz\mathbf{j} + cxyz^2\mathbf{k}$ When you have identified which of these forces is irrotational (i.e. has zero curl), you can find the potential function by calculating the work done when the force moves from the origin to (x, y, z) along any route you choose. indeed, you might try more than one route to convince yourself that the line integral is route-independent. One could devise many exercises in determining whether various force functions are conservative, and, if so, what the corresponding potential energy functions are, but i am going to restrict this chapter to just one more topic, namely 2. The Time and Energy Equation Consider a one-dimensional situation in which there is a force $F(x)$ that depends on the one coordinate only and is therefore a conservative force. If a particle moves under this force, its equation of motion is $m\ddot{x} = F(x)$ and we can obtain the space integral in the usual fashion by writing $\dot{x}$ as v . Thus $m v dv = F(x). dx$ 9.2.1 9.2.2 Integration yields $\frac{1}{2}mv^2 = F(x)x + T_0$ 9.2.3 Here $\frac{1}{2}mv^2$ is called the kinetic energy and the integration constant T_0 can be interpreted as the initial kinetic energy. Thus the gain in kinetic energy is $T - T_0 = \int_{x_0}^x F(x) dx$ 9.2.4

the right hand side merely being the work done by the force. Since F is a function of x alone, we can find a function V such that $F = -dV/dx$. [It is true that we could also find a function W such that $F = +dW/dx$, but we shall shortly find that the choice of the minus sign gives a desirable property that we can make use of.] If we integrate this equation, we find

$V = - \int F(x) dx + V_0$. 9.2.5 Here V is the potential energy and V_0 is the initial potential energy. From equations 9.2.4 and 9.2.5 we obtain $V + T = V_0 + T_0$. 9.2.6 Thus the quantity $V + T$ is conserved under the action of a conservative force. (This would not have been the case if we had chosen the $+$ sign in our definition of V .) We may call the sum of the two energies E , the total energy, and we have $T = E - V(x)$. 9.2.7 or $\frac{1}{2}mv^2 = E - V(x)$. 9.2.8 With $v = dx/dt$, we obtain, by integrating equation 9.2.8, $\int_{x_0}^x \frac{dx}{v} = \frac{1}{v_0} \int_{x_0}^x \frac{dx}{v}$. 9.2.9 This may at first appear to be a very formal and laborious way of arriving at something very obvious and something we have known since we first studied physics, but we shall see that it can often be a quite useful equation. You might, by the way, check that this equation is dimensionally correct. In some classes of problem such as pendulums, or rods falling over, the potential energy can be written as a function of an angle, and the kinetic energy is rotational kinetic energy written in the form $\frac{1}{2}I\omega^2$, where $\omega = d\theta/dt$. In that case, equation 9.2.9 takes the form $\int_{\theta_0}^{\theta} \frac{d\theta}{\omega} = \frac{1}{\omega_0} \int_{\theta_0}^{\theta} \frac{d\theta}{\omega}$. 9.2.10 You should check that this, too, is dimensionally correct.

3. Examples i. A body is projected vertically upward with initial speed v_0 . How long does it take to reach height y ? Of course you can do this by methods that you have known since high school, and using equation 9.2.9 might seem like using a sledgehammer to crack a nut, but it is worthwhile to start with a simple example, just to see how it works, before going on to other examples. 1 2 The total energy E is equal to the initial kinetic energy $\frac{1}{2}mv_0^2$ and the potential energy at height y is mgy , so that equation 9.2.9 becomes $\frac{1}{2}mv_0^2 - mgy = \frac{1}{2}mv^2$. 9.3.1 The result of the integration (which may cause a good deal more trouble than solving the problem by the methods you learned in high school) is $t = (v_0 - \sqrt{v_0^2 - 2gy})/g$. 9.3.2 To do the problem by high-school methods, you would probably just use the equation $y = v_0 t - \frac{1}{2}gt^2$. 9.3.3 Equation 9.3.2. is just one of the two solutions of this quadratic equation in t . (What is the significance of the other solution?) ii. What is the period of oscillation of a pendulum of length l swinging through an angle θ . The answer

is that it is four times the time that it takes to rise from the vertical position to an angle θ from the vertical. Thus we shall work out this time from equation 9.2.10 and multiply by four.

FIGURE IX. 1 Let us adopt the upper end of the string as our level for zero potential energy. The potential energy $V(\theta)$ is then $-mgl \cos \theta$. The total energy E is equal to the potential energy when $\theta = 0$ or. The rotational inertia I is ml^2 . Thus equation 9.2.10 gives $P = 4t = \frac{8}{\omega_0} \int_0^{\theta} \frac{1}{\sqrt{4\cos \theta - \cos \theta_0}} d\theta$ 9.3.4 This can doubtless be expressed in terms of special functions that most of us are unfamiliar with, so you will probably opt to evaluate this numerically as a function of θ . It would be of interest to calculate the period, in units of the familiar small-angle $2\pi/\omega_0$, as a function of θ_0 going from 0 to 90° . For example, if $\theta_0 = 90^\circ$, I make the period 1.18 times $2\pi/\omega_0$. I'm not going into the methods of numerical integration here, except to point out that, whatever method you use (including using pre-packed calculator programs) you will encounter a difficulty at the upper limit of the integration when the integrand becomes infinite. Using a trigonometric identity, you can write equation 9.3.4 as $\int_0^{\theta} \frac{1}{\sqrt{1 - \sin^2 \theta_0 \sin^2 \theta}} d\theta$ 9.3.5 This doesn't get rid of the infinity, but now make a change of variable by letting $\sin \theta = \sin \theta_0 \sin \phi$ 9.3.6

and the difficulty will disappear. You will notice that in all cases in which the system is at rest at either the start or the finish, the denominator of equation 9.2.9 or 9.2.10 will necessarily be zero and hence the integrand will be infinite. In some cases, as in example (i), the integral can be done analytically and there is no problem. If, however, as in example (ii), the integration has to be done numerically, there is a potential problem and some ingenuity (often by making a change of variable) will have to be exercised. iii. A Semicircle, a Ring and a String In this example, we have a smooth semicircular wire in a vertical plane. The radius of the semicircle is a . A ring of mass m at P can slide smoothly around the ring. An elastic string of natural length $2a$ is attached to the ends of the wire at A and B and is threaded through the ring. The force constant of the string is k . The ring is subjected to three conservative forces - gravity and the tensions in the two parts of the string. The ring is smooth, so there is no nonconservative friction. By geometry the lengths of the following are $AP: 2a \cos \theta$ $BP: 2a \sin \theta$ $PM: 2a \sin \theta \cos \theta = a \sin 2\theta$

We are going to find the equilibrium position(s) of the ring, and see how long it takes to slide from one position on the semicircle to another. Before doing any calculations, let's think about the physics qualitatively. Suppose that it is a very heavy ring and a weak string. In that case the ring will surely slide down to the bottom of the semicircle and stay there. On the other hand suppose that the ring is not very heavy but the string is quite strong. In that case we may well imagine that the ring may rest in stable equilibrium farther up the semicircle; indeed, if the string is very strong, the stable equilibrium position of the ring might be quite near the top. Of course, by the symmetry of the situation, there will always be an equilibrium position at the bottom of the semicircle, but, if the string is very strong, this position will be unstable, and, upon the slightest displacement, the ring will snap up to a higher position. Whether the position at the bottom of the semicircle will be stable or unstable depends on the relative strengths of the weight of the ring and the tension in the string. Let us then, in anticipation, refer to the ratio mg/ka by the symbol X . In fact we shall find that if $X > 0.586$, the position at the bottom of the semicircle will be stable, but if it is less than this the positions of stable equilibrium will be higher up. The extension of the string above its natural length is $2a(\cos\theta + \sin\theta - 1)$ and the depth of the ring below AB is $a\sin 2\theta$. Therefore, if we take AB to be the level for zero gravitational potential energy, the potential energy (elastic plus gravitational) is $V(\theta) = -k[2a(\cos\theta + \sin\theta - 1)]^2 - mga\sin 2\theta : ka^2[2(\cos\theta + \sin\theta - 1)^2 - X\sin 2\theta]$ 9.3.7

Figure IX.3 shows this potential energy as a function of θ for several values of X , including 0.586. We can see that, for X greater than this, the position at the bottom of the ring ($\theta = 45^\circ$) is the only equilibrium position and it is stable. For small values of X , the bottom, while an equilibrium position, is unstable, and there are two stable positions higher up. To calculate these equilibrium positions exactly, we need to determine where the derivative of V is zero, and to find whether these positions are stable or unstable, we need to examine the sign of the second derivative. I leave it to the reader to work through the first derivative and to show that one condition for the derivative to be zero is for $\cos\theta$ to equal $\sin\theta$; that is, $\theta = 45^\circ$, which corresponds to the bottom of the semicircle. Another condition for the first derivative to be zero is slightly more challenging to find, but you should find that the derivative is zero if

9.3.8

This corresponds to a real value of θ only if $X \leq 2^{-1/2} = 0.586$. The second

derivatives are necessary to determine whether the equilibria are stable (V a minimum) or unstable (V a maximum) or a glance at figure IX.3 will be easier. Now let's express the potential energy as a function U of the angle q , so that $U(q) = V(0)$. From Figure IX.2, we see that $q = 90^\circ - 20^\circ$. After some algebra and trigonometry, I find that the potential energy as a function of q is given by $U(q) = 4\cos^2 \frac{q}{2} - 4\cos \frac{q}{2} \cos 0 + 4\cos^2 0$. 9.3.9 Now expand this carefully by Taylor's theorem as far as q^2 : 9.3.10 What we have done is to approximate the potential energy function $U(q)$ by a parabola for small q . Now a parabolic potential well is characteristic of simple harmonic motion. For example, in linear simple harmonic motion obeying the equation $m\ddot{x} = -kx$ the potential energy per unit

mass is $\frac{1}{2}kx^2$ and the period is 2π . In rotational simple harmonic motion obeying the equation $I\ddot{\theta} = -c\theta$ the potential energy per unit rotational inertia is $\frac{1}{2}c\theta^2$ and the period is 2π . The rotational inertia here is just ma^2 , so we find that the period of small oscillations is $2\pi\sqrt{\frac{a^2}{g}}$ provided, of course, that $X > 2a$. 9.3.11 If you want to find how long the ring takes to slide from an initial position $q = \theta_0$ to the bottom you can use equations 9.2.11 and 9.3.10, with $E = U(0)$. You will find the usual difficulty that the integrand is zero at the start. I haven't actually tried the problem, because it looks slightly tedious, but I am fairly certain that it can be integrated analytically. If you do it and get an answer, make sure that, in the limit of small q , you get the same as equation 9.3.11. iii. How long would it take for a stone, falling from rest at a distance of 240,000 miles, to reach Earth? Let r be the initial distance of 240,000 miles, and let r_2 be the radius of Earth. The potential energy at distance r is $-GM/r$ and the total energy is the initial potential energy $-GM/r$. The time taken is therefore $\int_{r_2}^r \frac{dr}{\sqrt{2GM(\frac{1}{r} - \frac{1}{r_2})}}$. 9.3.12 $t = \sqrt{\frac{r_2^3}{GM}} \int_0^{\pi/2} \frac{d\phi}{\sin \phi}$ The gravity on the surface of Earth is $g = \frac{GM}{r_2^2}$ so that equation 9.3.12 can be written $t = \sqrt{\frac{r_2}{g}} \int_0^{\pi/2} \frac{d\phi}{\sin \phi}$. This can be easily integrated analytically (can't it? !) to give $t = \sqrt{\frac{r_2}{g}} \ln \left(\frac{1 + \sin \phi}{1 - \sin \phi} \right) \bigg|_0^{\pi/2}$. 9.3.13 $t = \sqrt{\frac{r_2}{g}} \ln 2$. 9.3.14 where $\cos 2\phi = r/r_2$. 9.3.15

In case that's too difficult, here's how to do it in your head in a few seconds. 240,000 is the radius of the Moon's orbit. The stone is falling in a highly elliptical orbit of major axis equal to the distance to the Moon - i.e. its semi major axis is half that of the Moon. Therefore by Kepler's third law, its period is equal to the Moon's period (which is 28 days) divided by $2^{3/2}$, which is 2.8. The orbital period of the stone is therefore 10 days. The time taken to

drop to Earth is half of this, or five days. v. A rod of length $2l$ and mass m has one end freely pivoted on a horizontal floor. The rod is held at an initial angle of 45° to the vertical and then released. How long does it take for the rod to hit the floor? I'll leave you to work this one out. By the way, if I had started with the rod vertical, you would find that it takes an infinite time to fall - because the vertical position, although unstable, is an equilibrium position, and it would never get going unless given an infinitesimal displacement.

9.4 Virtual Work

We have seen that a mechanical system subject to conservative forces is in equilibrium when the derivatives of the potential energy with respect to the coordinates are zero. A method of solving such problems, therefore, is to write down an expression for the potential energy and put the derivatives equal to zero. A very similar method is to use the principle of virtual work. In this method, we imagine that we act upon the system in such a manner as to increase one of the coordinates. We imagine, for example, what would happen if we were to stretch one of the springs, or to increase the angle between two jointed rods, or the angle that the ladder makes as it leans against the wall. We ask ourselves how much work we have to do on the system in order to increase this coordinate by a small amount. If the system starts from equilibrium, this work will be very small, and, in the limit of an infinitesimally small displacement, this "virtual work" will be zero. This method is very little different from setting the derivative of the potential energy to zero. I mention it here, however, because the concept might be useful in Chapter 13 in describing Hamilton's variational principle. Let's start by doing a simple ladder problem by the method of virtual work. The usual uniform ladder of high school physics, of length $2l$ and weight mg , is leaning in limiting static equilibrium against the usual smooth vertical wall and the rough horizontal floor whose coefficient of limiting static equilibrium is μ . What is the angle θ that the ladder makes with the vertical wall? I have drawn the four forces on the ladder, namely: its weight mg ; the normal reaction of the floor on the ladder, which must also be mg ; the frictional force, which is μmg ; and the normal (and only) reaction of the wall on the ladder, which must also be μmg .

There are several ways of doing this, which will be familiar to many readers. The only small reminder that I will give is to point out that, if you wish to combine the two forces at the foot of the ladder into a single force acting upwards and somewhat to the left, so that there are then just three forces

principle of virtual work. Ai gmg ,mg mg B D gmg E FIGURE IX.4 Before starting, I should warn that it is important in using the principle of virtual

8(BE) = 2l cos 0 80. 9.4.4 Further, if were to increase 0 by 80, the work done by the force at C would be mg times the decrease of the distance CD, and the

work done by the frictional force at E would be minus mg times the increase of the distance BE. The other two forces do no work. Thus the "virtual work"

done by the external forces on the ladder is $mg \cdot 1 \sin 0.80 - [tmg \cdot 21 \cos 0.80]$.
9.4.5 On putting the expression for the virtual work to zero, we obtain $\tan \theta =$

2g. 9.4.6 You should verify that this is the same answer as you get from other methods - the easiest of which is probably to take moments about E. There is

something about virtual work which reminds me of thermodynamics. The first law of thermodynamics, for example is $\Delta U = \Delta q + \Delta w$, where ΔU is the

increase of the internal energy of the system, A_q is the heat added to the system, and A_w is the work done on the system. Prepositions play an

important part in thermodynamics. It is always mandatory to state clearly and without ambiguity whether work is done by the piston on the gas, or by the

gas on the system; or whether heat is gained by the system or lost from it. Without these prepositions, all discussion is meaningless. Likewise in solving

a problem by the principle of virtual work, it is always essential to say whether you are describing the work done by a force on what part of the

system (on the ladder or on the floor?) and whether you are describing an increase or a decrease of some length or angle. Let us move now to a slightly

more difficult problem, which we'll try by three different methods - including that of virtual work. In figure IX.5, a uniform rod AB of weight Mg and

length $2a$ is freely hinged at A. The end B carries a smooth ring of negligible

mass. A light inextensible string of length l has one end attached to a fixed point C at the same level as A and distant $2a$ from it. It passes through the ring and carries at its other end a weight Mg hanging freely. (The "smooth" ring means that the tension in the string is the same on both sides of the ring.) Find the angle CAB when the system is in equilibrium. I have marked in various angles and lengths, which can easily be determined from the geometry of the system, and I have also marked the four forces on the rod.

R D $2a$ A B O $2a \cos \theta$ Mg $2a \sin \theta$ $4a \sin \theta$ C Mg Mg $l - 4a \sin \theta$

FIGURE IX.5 Let us first try a very conventional method. We know rather little about the force R of the hinge on the rod (though see below), and therefore this is a good reason for taking moments about the point A . We immediately obtain $Mg a \cos \theta + Mg \cdot 2a \cos \theta = Mg \cdot 2a \cos \theta$. 9.4.7

Divide by Mga and set $\cos \theta = 2c^2 - 1$, where $c = \cos 30$. After a little algebra, we obtain $12c^2 - c - 6 = 0$ and hence we find for the equilibrium condition that $\theta = 82^\circ 49'$ or $263^\circ 37'$. The latter, by the way, is a physically valid solution - you might want to sketch it. I pointed out that the fact that we knew little about the reaction R at the hinge was a good reason for taking moments about A - although in fact I have drawn R in about the right direction. If you replace the two forces at the ring with a single force, this single force will bisect the angle between the two portions of the string. There are then three forces on the rod, and they must be concurrent at a point. That's how I knew the direction of R . However, now that we know some angles, it would be a simple matter, if need be, to find the horizontal and vertical components of R . Now let's try the same problem using energy conditions. We'll take the zero of potential energy when the rod is horizontal - at which time the small mass is at a distance l below the level AC . When the angle $CAB = \theta$, the distance of the centre of mass of the rod below AC is $a \sin \theta$ and the distance of the small mass below AC is $l - 4a \sin \frac{1}{2} \theta + 2a \sin \theta$, so that the potential energy is $V = -Mga \sin \theta + \frac{1}{2} Mg[l - (l - 4a \sin \frac{1}{2} \theta + 2a \sin \theta)] = -Mga(3 \sin \frac{1}{2} \theta - \sin \theta)$. 9.4.8

$dV/d\theta$ The derivative is $- : Mga(3 \cos \frac{1}{2} \theta - \cos \theta)$, 9.4.9

$dV/d\theta = 0$ and setting this to zero will produce the same results as before. Further differentiation (do it), or a graph of V' (do it), will show that the $82^\circ 49'$ solution is stable and the $263^\circ 37'$ solution is unstable. Now let's try it by virtual work. We are going to increase θ by $\delta \theta$ and see how much work is done. The distance of the centre of mass of the rod below AC is $a \sin \theta$, and if θ

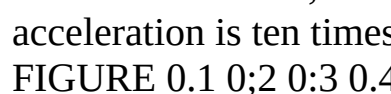
increases by θ , this will increase by $a \cos \theta$, and the work done by Mg will be $Mga \cos \theta$. The distance of the ring below AC is $2a \sin \theta$, and if θ increases by $\delta\theta$, this will increase by $2a \cos \theta \delta\theta$, and the work done by the downward force will be $-Mg \cdot 2a \cos \theta \delta\theta$. The distance BC is $4a \sin \theta$, and if θ increases by $\delta\theta$, this will increase by $4a \cos \theta \delta\theta$, and the work done by the sloping force will be $-\frac{1}{2}Mg \cdot 4a \cos \theta \delta\theta$. Thus the virtual work is

$Mga \cos \theta \delta\theta - Mg \cdot 2a \cos \theta \delta\theta - \frac{1}{2}Mg \cdot 4a \cos \theta \delta\theta$. If we put this equal to zero, we obtain the same result as before. 9.4.10

1. Introduction 1 CHAPTER 10 ROCKET MOTION If you are asked to state Newton's Second Law of Motion, I hope you will not reply: "Force equals mass times acceleration" - because that is not Newton's Second Law of Motion. Newton's Second Law of Motion is: 'The alteration of motion is ever proportional to the motive force impressed,' and is made in the direction of the right line in which that force is impressed. In short: Force equals Rate of Change of Momentum, or, in symbols, $F = \frac{dp}{dt}$. On differentiating the right hand side, we obtain $F = m \frac{dv}{dt} + v \frac{dm}{dt}$. In other words, if the mass is constant then indeed force equals mass times acceleration - but only if the mass is constant. In a rocket, a very appreciable fraction of the mass of the rocket is fuel, which is burned and ejected at a very high rate, so that the mass of the rocket is rapidly diminishing during the motion. It is one of the great problems of rocket design that such a high proportion of the initial mass must be fuel. For this reason, other possible methods of driving spacecraft are being investigated by many groups. For example, in the ion propulsion system of the Deep Space One spacecraft, electrically accelerated ions are ejected at high speed from the spacecraft. The force produced and the acceleration are minute, but, because it can be kept up for a very long time, very high speeds can eventually be reached. "Solar sail" systems similarly rely on the very tiny force that can be exerted by the solar wind, but this tiny force can be exerted during most of the lifetime of a spacecraft's flight, and hence again high speeds can be reached. This chapter, however, concerns just conventional rocket motion. In the next section I consider the motion of a rocket in space subject only to the one force from the high-speed ejection of burned fuel in the absence of any other forces. At a later date, if I can find the time and energy, I may add further sections on rocket motion against gravity, which might be uniform or might fall off with distance from Earth, and we might

include air resistance or not. But to begin with, we deal solely with a rocket isolated in space and subject to no additional forces. An Integral So that we don't get bogged down later with an integral that is going to crop up, see if you can do the following integration: $\int \ln(a - bt) dt = \frac{1}{b} \ln(a - bt) + \text{constant}$. You should get $-\frac{1}{b} \ln(a - bt) + \text{constant}$.

3. The Rocket Equation. Initially at time $t = 0$, the mass of the rocket, including fuel, is m_0 . We suppose that the rocket is burning fuel at a rate of b kg/s - so that, at time t , the mass of the rocket-plus-remaining-fuel is $m = m_0 - bt$. The rate of increase of mass with time is $dm/dt = -b$ and is supposed constant with time. (The rate of "increase" is, of course, negative.) We suppose that the speed of the ejected fuel, relative to the rocket, is v . The thrust of the ejected fuel on the rocket is therefore vb , or $-v dm/dt$. This is equal to the instantaneous mass times acceleration of the rocket: $dv = (m_0 - bt)^{-1} dv vb = m^{-1} dt$. 10.3.1 Thus $\int dv = \int \frac{vb}{m_0 - bt} dt$ 10.3.2 $m_0 - bt = i$ dt You are anticipating a (Don't be tempted to write the right hand side as $-vb \ln(m_0 - bt)$, so keep the denominator positive. We have met this before in Chapter 6.) On integration, we obtain $v = v \ln \frac{m_0}{m_0 - bt}$ 10.3.3 $m_0 - bt$ The acceleration is $\frac{dv}{dt} = \frac{vb}{m_0 - bt}$ 10.3.4 At $t = 0$, the speed is zero and the acceleration is v/m_0 . At time $t = m_0/b$, the remaining mass is zero and the speed and acceleration are both infinite. However, this is so only if the initial mass is 100% fuel and nothing else. This is not realistic. If the fraction of the total mass was initially f , the fuel will be completely expended after a time $f m_0/b$ at which time the speed will be $-v \ln(1-f)$ (which is, of course, positive), and the speed will remain constant thereafter. For example, if $f = 99\%$, the final speed will be $4.6 v$.

Equations 10.3.3 and 10.3.4 are shown in figures X. 1 and X.2. In figure X. 1, the speed of the rocket is plotted in units of v , the ejection speed of the burnt fuel. The time is plotted in units of m_0/b . The fuel initially comprised 90% of the rocket, so that the rocket runs out of fuel in time $0.9 m_0/b$, at which time its speed is $2.3v$. In figure X.2, the acceleration is plotted in units of the initial acceleration, which is v/m_0 . When the fuel is exhausted, the acceleration is ten times this.  IURE x, 15' 0.. :5, 0;6 O:? 0:8 0':: thil
FIGURE 0.1 0;2 0:3 0.4 0 5 06 0? 08 019 time

In equation 10.3.3, t is of course dx/dt , so the equation can be integrated to

obtain the distance:time relation: 10.3.5 Elimination of f between equations 10.3.3 and 10.3.5 gives the relation between speed and distance: 10.3.6 If f is the fraction of the initial mass that is fuel, the fuel supply will be exhausted after a time $f m_0/b$, at which time its speed will be $-V \ln(1 - f)$, (this is positive, because $1 - f$ is less than 1), its acceleration will be $1/(1 - f)$ and it will have travelled a distance $V m_0/b [f + (1 - f) \ln(1 - f)]$. If the entire initial mass is fuel, so that $f = 1$, the fuel will burn for a time m_0/b , at which time its speed and acceleration will be infinite, it will have travelled a finite distance $V m_0/b$ and the mass will have been reduced to zero, This remarkable result is not very believable, for two reasons. In the first place it is not very realistic. More importantly, when the speed becomes comparable to the speed of light, the equations which we have developed for nonrelativistic speeds are no longer approximately valid, and the correct relativistic equations must be used. The speed cannot then reach the speed of light as long as the remaining mass is non-zero. Equations 10.3.5 and 10.3.6 are illustrated in figures X.3 and X.4, in which f , the fraction of the initial mass that is fuel, is 0.9. The units for distance, time and speed in these graphs are, respectively, $V m_0/b$, m_0/b and V .

0.5 FIGURE X.4 distance vs. time, 0.7 FIGURE X.3 speed vs. time
0.5 time

Problems (i) Derive the integral in section 2. (ii) Integrate equation 10.3.2 to obtain equation 10.3.3 (iii) Integrate equation 10.3.3 to obtain equation 10.3.5 (iv) Obtain equation 10.3.6 In the following problems, (numbers v - viii) assume $V = 2 \text{ km s}^{-1}$, $m_0 = 2000 \text{ kg}$, $b = 0.5 \text{ kg s}^{-1}$, $f = 90\%$ (v) (vi) (vii) (viii) What is the maximum speed, and how long does it take to attain it? How long does it take to reach a speed of 3 km s^{-1} ? How long does it take for the rocket to travel 600 km ? How fast is it moving when it has travelled 300 km ?

1 CHAPTER 11 SIMPLE AND DAMPED OSCILLATORY MOTION 11.1 Simple Harmonic Motion I am assuming that this is by no means the first occasion on which the reader has met simple harmonic motion, and hence in this section I merely summarize the familiar formulas without spending time on numerous elementary examples Simple harmonic motion can be defined as follows: It a point P moves in a circle of radius a at constant angular speed

co (and hence period $2\pi/\omega$) its projection Q on a diameter moves with simple harmonic motion. This is illustrated in figure XI. 1, in which the velocity and acceleration of P and of Q are shown as coloured arrows. The velocity of P is just ωa and its acceleration is the centripetal acceleration $\omega^2 a$. As in Chapter 8 and elsewhere, I use blue arrows for velocity vectors and green for acceleration. P0

FIGURE XI. 1

2 P0 is the initial position of P - i.e. the position of P at time $t = 0$ - and ϕ is the initial phase angle. At time t later, the phase angle is $\omega t + \phi$. The projection of P upon a diameter is Q. The displacement of Q from the origin, and its velocity and acceleration, are $y = a \sin(\omega t + \phi)$ 11.1.1 $v = \omega a \cos(\omega t + \phi)$ 11.1.2 $a = -\omega^2 a \sin(\omega t + \phi)$ 11.1.3 Equations 11.1.2 and 11.1.3 can be obtained immediately either by inspection of figure XI. 1 or by differentiation of equation 11.1.1. Elimination of the time from equations 11.1.1 and 11.1.2 and from equations 11.1.1 and 11.1.3 leads to $v^2 = \omega^2(a^2 - y^2)$ 11.1.4 $y = \frac{1}{\omega} \sqrt{a^2 - v^2}$ 11.1.5 and An alternative definition of simple harmonic motion is to define as simple harmonic motion any motion that obeys the differential equation 11.1.5. We then have the problem of solving this differential equation. We can make no progress with this unless we remember to write $y = a \sin(\omega t + \phi)$ as dy/dt (recall that we did this often in Chapter 6.) Equation 11.1.5 then immediately integrates to $v^2 = \omega^2(a^2 - y^2)$. 11.1.6 A further integration, with $v = dy/dt$, leads to $y = a \sin(\omega t + \phi)$, 11.1.7 provided we remember to use the appropriate initial conditions. Differentiation with respect to time then leads to equation 11.1.2, and all the other equations follow. Important Problem. Show that $y = a \sin(\omega t + \phi)$ can be written $y = A \sin \omega t + B \cos \omega t$, 11.1.8 where $A = a \cos \phi$ and $B = a \sin \phi$. The converse of these are $A = y \cos \phi + B \sin \phi$, $B = y \sin \phi - A \cos \phi$. It is important to note that, if A and B are known, in order to calculate ϕ without ambiguity of quadrant it is entirely necessary to calculate

3 both $\cos \phi$ and $\sin \phi$. It will not do, for example, to calculate ϕ solely from $\phi = \tan^{-1}(y/x)$, because this will give two possible solutions for ϕ differing by 180° . Show also that equation 11.1.8 can also be written $y = M e^{i\omega t} + N e^{-i\omega t}$, 11.1.9 where $M = (B - iA)/2$ and $N = (B + iA)/2$ and show that the right hand side of equation 11.1.9 is real. The four large satellites of Jupiter furnish a beautiful demonstration of simple harmonic motion. Earth is almost in the plane of their orbits, so we see the motion of satellites projected on a

diameter. They move to and fro in simple harmonic motion, each with different amplitude (radius of the orbit), period (and hence angular speed ω) and initial phase angle ϕ .

11.2 Mass Attached to an Elastic Spring.

I am thinking of a mass m resting on a smooth horizontal table, rather than hanging downwards, because I want to avoid the unimportant distraction of the gravitational force (weight) acting on the mass. The mass is attached to one end of a spring of force constant k , the other end of the spring being fixed, and the motion is restricted to one dimension. I suppose that the force required to stretch or compress the spring through a distance x is proportional to x and to no higher powers; that is, the spring obeys Hooke's Law. When the spring is stretched by an amount x there is a tension kx in the spring; when the spring is compressed by x there is a thrust kx in the spring. The constant k is the force constant of the spring. When the spring is stretched by an distance x , its acceleration a is given by $ma = -kx$.

11.2.1

This is an equation of the type 11.1.5, with $\omega^2 = k/m$, and the motion is therefore simple harmonic motion of period $P = 2\pi/\omega = 2\pi\sqrt{m/k}$.

11.2.2

At this stage you should ask yourself two things: Does this expression have dimensions T? Physically, would you expect the oscillations to be slow for a heavy mass and a weak spring? The reader might be interested to know (and this is literally true) that when I first typed equation 11.2.2, I inadvertently typed $\omega^2 = k/m$, and I immediately spotted my mistake by automatically asking myself these two questions. The reader might also like to note that you can deduce that $P \propto \sqrt{m/k}$ -- by the method of dimensions, although you cannot deduce the proportionality constant 2π . Try it.

4 Energy Considerations.

The work required to stretch (or compress) a Hooke's law spring by x is $\frac{1}{2}kx^2$, and this can be described as the potential energy or the elastic energy stored in the spring. I shall not pause to derive this result here. It is probably already known by the reader, or s/he can derive it by calculus. Failing that, just consider that, in stretching the spring, the force increases kx and so the work done is linearly from 0 to kx , so the average force used over the distance x is $\frac{1}{2}kx$. If we assume that, while the mass is oscillating, no mechanical energy is dissipated as heat, the total energy of the system at any time is the sum of the elastic energy $\frac{1}{2}kx^2$ stored in the spring and the kinetic energy $\frac{1}{2}mv^2$ of the mass. (I am assuming that the mass of the spring is negligible compared with m .)

Thus $E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$: 11.2.3 and there is a continual exchange of energy between elastic

and kinetic. When the spring is fully extended, the kinetic energy is zero and the total energy is equal to the elastic energy then, $\frac{1}{2} k a^2$; when the spring is unstretched and uncompressed, the energy is entirely kinetic; the mass is then moving at its maximum speed $a \omega$ and the total energy is equal to the kinetic energy then, $\frac{1}{2} m a^2 \omega^2$. Any of these expressions is equal to the total energy:

11.2.4 11.3 Torsion Pendulum A torsion pendulum consists of a mass of rotational inertia I hanging by a thin wire from a fixed point. If we assume that the torque required to twist the wire through an angle θ is proportional to θ and to no higher powers, then the ratio of the torque to the angle is called the torsion constant c . It depends on the shear modulus of the material of which the wire is made, is inversely proportional to its length, and, for a wire of circular cross-section, is proportional to the fourth power of its diameter. A thick wire is much harder to twist than a thin wire. Ribbonlike wires have comparatively small torsion constants. The work required to twist a wire through an angle θ is $\frac{1}{2} c \theta^2$. When a torsion pendulum is oscillating, its equation of motion is $I \ddot{\theta} = -c \theta$.

11.3.1 This is an equation of the form 11.1.5 and is therefore simple harmonic motion in which $\omega = \sqrt{c/I}$. This example, incidentally, shows that our second definition of simple harmonic motion (i.e. motion that obeys a differential equation of the form of equation 11.1.5) is a more general definition than our introductory description as the projection upon a diameter of uniform motion in a circle. In particular, do not imagine that ω here is the same thing as ω !

Exercise: Write down the torsional analogues of all the equations given for linear motion in sections 11.1 and 11.2.

11.4 Ordinary Homogeneous Second-order Differential Equations. This is not a full mathematical course on differential equations, but it may be useful as a reminder for those who have already studied differential equations, and may even be just enough for our purposes for those who have not. We suppose that $y = y(x)$ and y' denotes dy/dx . An ordinary homogeneous second-order differential equation is an equation of the form $ay'' + by' + cy = 0$, 11.4.1 and we have to find a function $y(x)$ which satisfies this. It turns out that it is quite easy to do this, although the nature of the solutions depends on whether b^2 is less than, equal to or greater than $4ac$. A first point to notice is that, if $y = f(x)$ is a solution, so is $Af(x)$ - just try substituting this in the equation 11.4.1. If $y = g(x)$ is another solution, the same is true of g - i.e. $Bg(x)$ is also a solution. And you can also easily verify that any linear combination, such as $y = As(x) + Bs(x)$, 11.4.2 is

also a solution. Now equation 11.4.1 is a second-order equation - i.e. the highest derivative is a second derivative - and therefore there can be only two arbitrary constants of integration in the solution - and we already have two in equation 11.4.1, and consequently there are no further solutions. All we have to do, then, is to find two functions that satisfy the differential equation. It will not take long to discover that solutions of the form $y = e^{kx}$ satisfy the equation, because then $y' = ky$ and $y'' = k^2y$, and, if you substitute these in equation 11.4.1, you obtain $(k^2 + bk + c)y = 0$. 11.4.3 You can always find two values of k that satisfy this, although these may be complex, which is why the nature of the solutions depends on whether b^2 is less than or greater than $4ac$. Thus the general solution is $y = e^{k_1x} + Be^{k_2x}$ 11.4.4 where k and k_2 are the solutions of the equation

$6ax + bx + c = 0$. 11.4.5 There is one complication, however, if $b^2 = 4ac$, because then the two solutions of equation 11.4.5 are each equal to $-b/(2a)$. The solution of the differential equation is then $y = (C + Bx)\exp[-bx/(2a)]$ 11.4.6 and the two constants can be combined into a single constant $C = C + B$ so that equation 11.4.6 can be written $y = C\exp[-bx/(2a)]$. 11.4.7 This solution has only one independent arbitrary constant, and so an additional solution must be possible. Let us try and see whether a function of the form $y = x\exp[-bx/(2a)]$ 11.4.8 might be a solution of equation 11.4.1 From equation 11.4.8 we obtain $y' = (1 - mx)\exp[-bx/(2a)]$ and $y'' = m(2 - mx)\exp[-bx/(2a)]$. On substituting these into equation 11.4.1, remembering that $c = b^2/(4a)$, we obtain for the left hand side of equation 11.4.1, after some algebra, $\exp[-bx/(2a)] [(2am + b)x + 4a(2am + b)]$. 11.4.9 This is identically zero if $m = -b/(2a)$, and hence $y = x\exp[-bx/(2a)]$ 11.4.10 is a solution of equation 11.4.1. the general solution of equation 11.4.1, if $b^2 = 4ac$, is therefore $y = (C + Dx)\exp[-bx/(2a)]$. 11.4.11 We shall discover what these solutions actually look like in the next section. 11.5 Damped Oscillatory Motion As pointed out in section 11.2, the equation of motion for a mass m vibrating on the end of a spring of force constant k , in the absence of any damping, is $m\ddot{x} = -kx$. 11.5.1 Here, I am assuming that the displacement x is a function of time, and a dot denotes d/dt .

However, in most real situations, there is some damping, or loss of mechanical energy, which is dissipated as heat. In the case of our example of a mass oscillating on a horizontal table, damping may be caused by friction between the mass and the table. For a mass hanging vertically from a spring,

we might imagine the mass to be immersed in a viscous fluid. These are obvious examples. Slightly less obvious, it may be that the constant bending and stretching of the spring produces heat, and the motion is damped from this cause. In any case, in this analysis we shall assume that, in addition to the restoring force kx , there is also a damping force that is proportional to the speed at which the particle is moving. I shall denote the damping force by $b\dot{x}$. The equation of motion is then $m\ddot{x} = -kx - b\dot{x}$. 11.5.2 For k/m and b/m , we obtain the equation of motion in its usual form. If I divide by m and write $\omega_0^2 = k/m$ and $\gamma = b/m$, we obtain the equation of motion in its usual form $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$. 11.5.3 Here γ is the damping constant, which we have already met in Chapter 10, and, from Section 11.4, we are ready to solve the differential equation 11.5.3. Indeed, we know that the general solution is $x = A e^{k_1 t} + B e^{k_2 t}$, 11.5.4 where k_1 and k_2 are the solutions of the quadratic equation $x^2 + \gamma x + \omega_0^2 = 0$. 11.5.5 An exception occurs if $k_1 = k_2$, and we shall deal with that exceptional case in due course (subsection 11.5.iii). Otherwise, k_1 and k_2 are given by $-\gamma/2 \pm i\sqrt{\omega_0^2 - \gamma^2/4}$. In section 11.5.3, we pointed out that the nature of the solution depends on whether γ^2 is less than, equal to or greater than $4\omega_0^2$, or, in the present case, upon whether γ is less than, equal to or greater than $2\omega_0$. These cases are referred to, respectively, as lightly damped, critically damped and heavily damped. We shall start by considering light damping. 11.5i Light damping: $\gamma < 2\omega_0$ Since $\gamma < 2\omega_0$, we have to write equations 11.5.6 as $-\gamma/2 \pm i\sqrt{\omega_0^2 - \gamma^2/4}$. 11.5.7 Further, I shall write $\omega = \sqrt{\omega_0^2 - \gamma^2/4}$. 11.5.8

Equation 11.5.4 is therefore $A e^{(-\gamma/2 + i\omega)t} + B e^{(-\gamma/2 - i\omega)t}$. If x is to be real, A and B must be complex. the asterisk denotes the complex conjugate. Also, since $e^{i\theta} = (e^{-i\theta})^*$, Let $A = (a - ib)$ and $B = (a + ib)$, where a and b are real. show that equation 11.5.9 can be written as $x = e^{-\gamma t/2} (a \cos \omega t + b \sin \omega t)$. And if $C = J - b^2$, $\sin \omega t = \cos(\omega t - \phi)$, $\cos \omega t = \sin(\omega t + \phi)$. 11.5.9 B must equal A^* , where $\phi = \tan^{-1}(b/a)$. Then the reader should be able to 11.5.10, the equation can be written $x = C e^{-\gamma t/2} \sin(\omega t + \phi)$. 11.5.11 Equations 11.5.10, 11 or 12 are three equivalent ways of writing the solution. Each has two arbitrary integration constants (A, B) , (a, b) or (C, ϕ) , whose values depend on the initial conditions - i.e. on the values of x and $\dot{x}$ when $t = 0$. Equation 11.5.11 shows that the motion is a sinusoidal oscillation of period a little less than $2\pi/\omega$, with an exponentially decreasing amplitude. To find C and ϕ in terms of the initial conditions, differentiate equation 11.5.11 with respect to the time in order to obtain an equation showing the speed as a function of the time: $\dot{x} = C e^{-\gamma t/2} [\omega \cos(\omega t + \phi) - \gamma/2 \sin(\omega t + \phi)]$.

or)]. 11.5.12 By putting $t = 0$ in equations 11.5.11 and 11.5.12 we obtain $x_0 = C \sin \phi$ 11.5.13 and $(\dot{x})_0 = C(\omega' \cos \phi - \gamma \sin \phi)$. 11.5.14 From these we easily obtain $\cot \phi = \omega' / \gamma$ 11.5.15 $C = x_0 \csc \phi$ 11.5.16

The quadrant of ϕ can be determined from the signs of $\cot \phi$ and $\csc \phi$, C always being positive. Note that the amplitude of the motion falls off with time as $e^{-\gamma t}$, but the mechanical energy, which is proportional to the square of the amplitude, falls off as $e^{-2\gamma t}$. Figures XI.2 and XI.3 are drawn for $C = 1$, $\phi = 0$, $\gamma = 1$. Figure XI.2 has $\omega' = 257$ and hence $x = e^{-t} \sin 257t$, and figure XI.3 has $\omega' = 47$ and hence $x = e^{-t} \sin 47t$. 11.5.17 11.5.18

FIGURE XI.2 1 2 3 4 5 6 t Problem. Draw displacement' time graphs for an oscillator with $m = 0.02$ kg, $k = 0.08$ N m⁻¹, $\gamma = 1.5$ s⁻¹, $t = 0$ to 15 s, for the following initial conditions: i. $x_0 = 0$, $(\dot{x})_0 = 4$ m s⁻¹ ii. $(\dot{x})_0 = 0$, $x_0 = 3$ m iii. $(\dot{x})_0 = -2$ m s⁻¹, $x_0 = 2$ m

Although the motion of a damped oscillator is not strictly "periodic", in that the motion does not repeat itself exactly, we could define a "period" $P = 2/\omega'$ as the interval between two consecutive ascending zeroes. Extrema do not occur exactly halfway between consecutive zeroes, and the reader should have no difficulty in showing, by differentiation of equation 11.5.11, that extrema

occur at times given by $\tan(\omega' t + \phi) = 2\gamma / \omega'$. large, consecutive extrema occur approximately at intervals of $P/2$. maximum displacements is, then, 10

However, provided that the damping is not very large The ratio of consecutive From this, we find that the logarithmic decrement is $\ln(I_{x,1} / I_{x,2}) = \gamma P$, from which the damping constant can be determined. 11.5.17 11.5.18

FIGURE XI.3 0.8 0.6 0.4 x 0.2 0 -0.2 -0.4 -0.6 0 6 1 2 3 4 5 11.5ii Heavy damping: $\gamma > 2\omega'$ The motion is given by equations 11.5.4 and 11.5.6 where, this time, k_1 and k_2 are each real and negative. For convenience, I am going to write $X_1 = -k_1$ and $X_2 = -k_2$. and 2 are both real and positive, with $2 > k_1$ given by

11 The general solution for the displacement as a function of time is $x = X_1 e^{X_1 t} + X_2 e^{X_2 t} + X_3 e^{-\gamma t} \sin \omega' t + X_4 e^{-\gamma t} \cos \omega' t$. The speed is given by $\dot{x} = X_1 e^{X_1 t} + X_2 e^{X_2 t} - \gamma X_3 e^{-\gamma t} \sin \omega' t + \omega' X_3 e^{-\gamma t} \cos \omega' t - \gamma X_4 e^{-\gamma t} \cos \omega' t - \omega' X_4 e^{-\gamma t} \sin \omega' t$. 11.5.19

11.5.20 11.5.21 The constants A and B depend on the initial conditions. Thus: $x_0 = A + B$ 11.5.22 and $(\dot{x})_0 = -\gamma(A + B) + \omega'(A - B)$. 11.5.23 From these, we obtain t : $(\dot{x})_0 + \gamma x_0 = \omega'(A - B)$, 11.5.24 Example 1. $x_0 = 5$, $(\dot{x})_0 = 0$. $x = 5 e^{-\gamma t} \cos \omega' t$

Figure XI.4 shows $x(t)$ for $x_0 = 1\text{ m}$, $Z = 1\text{ s}^{-2}$, $Z^2 = 2\text{ s}^{-2}$. The displacement will fall to half of its initial value at a time given by putting $x/x_0 = 1/2$ in equation 11.5.25. In this will in general require a numerical solution. In our example, however, the equation $1 = 2e^{-t/2}$ reduces to $e^{-t/2} = 1/2$, and if we let $u = e^{-t/2}$, this becomes $u^2 - 2u + 1 = 0$. The two solutions of this are $u = 1.707107$ or 0.292893 . The first of these gives a negative t , so we want the second solution, which corresponds to $t = 1.228$ seconds. The velocity as a function of time is given by 11.5.26. This is always negative. In figure XI.5, is shown the speed, which is $|\dot{x}|$, as a function of time, for our numerical example. Those who enjoy differentiating can show that the maximum

speed is reached in a time $\ln(L/L_0)$, and that the maximum speed, reached at $t = \ln 2 = 0.6931$ seconds, is 0.5 m s^{-1} . In our speed example, is the 1 0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2 0.1 0 0. FIGURE XI.4 t seconds 0.5 0.4 0.3 0.2 0.1 0 0 FIGURE XI.6 $|\dot{x}|$ t seconds

Example 2. $x_0 = 0$, $F_0 = F(>0)$. In this case it is easy to show that $x \rightarrow 0$ as $t \rightarrow \infty$. It is left as an exercise to show that x reaches a maximum value of F/k when $t = \sqrt{2F/k}$. Figure XI.6 illustrates equation 11.5.26 for $k = 1\text{ s}^{-2}$, $Z^2 = 2\text{ s}^{-2}$, $F = 5\text{ m s}^{-2}$. The maximum displacement of 1.25 m is reached when $t = \sqrt{2} = 0.6831\text{ s}$. It is also left as an exercise to show that equation 11.5.26 can be written $2Fe^{-t/2} = V \sinh(t/2)$. Figure XI.6 illustrates equation 11.5.27 for $V = 1.2$. FIGURE XI.6 x t seconds

Example 3. $x_0 = 0$, $F_0 = -F$. This is the really exciting example, because the suspense-filled question is whether the particle will shoot past the origin at some finite time and then fall back to the origin; or whether it will merely tamely fall down asymptotically to the origin without ever crossing it. The tension will be almost unbearable as we find out. In fact, I cannot wait; I am going to plot x versus t in figure XI.7 for $Z = 1\text{ s}^{-2}$, $Z^2 = 2\text{ s}^{-2}$, $x_0 = 1\text{ m}$, and three different values of V , namely 1, 2 and 3 m s^{-1} . FIGURE XI.7 x t seconds We see that if $V = 3\text{ m s}^{-1}$ the particle overshoots the origin after about 0.7 seconds. If $V = 1\text{ m s}^{-1}$, it doesn't look as though it will ever reach the origin. And if $V = 2\text{ m s}^{-1}$, I'm not sure. Let's see what we can do. We can find out when it crosses the origin by

putting $x = 0$ in equation 11.5.20, where A and B are found from equations 11.5.24 with $(0) = -V$. This gives, for the time when it crosses the origin, $t = \frac{1}{\omega} \ln \frac{V}{V - 2X_0}$ 11.5.28

15 Since $L_2 > L_1$, this implies that the particle will overshoot the origin if $V > L_2 x_0$, and this in turn implies that, for a given V , it will overshoot only if $2V^2/X_0^2 + (\omega^2 - \gamma^2) < 11.5.29$ V/X_0 For our example, $L_2 x_0 = 2 \text{ m s}^{-1}$, so that it just fails to overshoot the origin if $V = 2 \text{ m s}^{-1}$. For $V = 3 \text{ m s}^{-1}$, it crosses the origin at $t = \ln 2 = 0.6931 \text{ s}$. In order to find out how far past the origin it goes, and when, we can do this just as in Example 2. I make it that it reaches its maximum negative displacement of -0.125 m at $t = \ln 4 = 1.386 \text{ s}$. 11.5iii Critical damping: $\gamma = 2\omega$ Before embarking on this section, you might just want to refresh your memory of differential equations as described in section 11.4. In this case, L_1 and L_2 are each equal to $\gamma/2$. As discussed in section 4, the general solution is of the form $x = Ce^{-\gamma t} + Dte^{-\gamma t}$, 11.5.30 which can also be written in the form $x = Ce^{-\gamma t}(1 + at)$. 11.5.31 Either way, there are two arbitrary constants, which can be determined by the initial values of the displacement and speed. It is easy to show that $C = x_0$ and $a = \frac{1}{x_0} \left(\frac{dx}{dt} \right)_0$. 11.5.32 x_0 The particle will not go through zero unless $V \left(\frac{1}{x_0} \left(\frac{dx}{dt} \right)_0 + \gamma \right) > 3$. 11.5.33 I'll leave it to the reader to draw a graph of equation 11.5.19. Ideally the hydraulic door closer that you see near the tops of doors in public buildings should be critically damped. This will cause the door to close fastest without slamming. And we have already used the physics of impulsive forces in problem 2.1 of Chapter 8 to work out where to place a door stop for minimum reaction on the hinges. Truly an understanding of physics is of enormous importance in achieving the task of closing a door! A more subtle example is in the design of a moving-coil ammeter. In this instrument, the electric current is passed through a coil between the poles of a magnet, and the coil then swings around against the restoring force of a little spiral spring. The coil is wound on a light aluminium frame

16 called a former, and, as the coil (and hence the former) moves in the magnetic field, a little current is induced in the former, and this damps the motion of the coil. In order that the coil and the pointer should move to the equilibrium position in the fastest possible time without oscillating, the system should be critically damped - which means that the rotational inertia and the electrical resistance of the little aluminium former has to be carefully

designed to achieve this. 11.6 Electrical Analogues A charged capacitor of capacitance C is connected in series with a switch and an inductor of inductance L . The switch is closed, and charge flows out of the capacitor and hence a current flows through the inductor. Thus while the electric field in the capacitor diminishes, the magnetic field in the inductor grows, and a back electromotive force (EMF) is induced in the inductor. Let Q be the charge in the capacitor at some time. The current I flowing from the positive plate is equal to $-dQ/dt$. The potential difference across the capacitor is Q/C and the back EMF across the inductor is $L dI/dt = -L d^2Q/dt^2$. The potential drop around the whole circuit is zero, so that $Q/C = -L d^2Q/dt^2$. The charge on the capacitor is therefore governed by the differential equation which is simple harmonic motion with $\omega^2 = 1/LC$. You should verify that this has dimensions T^{-1} . If there is a resistor of resistance R in the circuit, while a current flows through the resistor there is a potential drop $RI = -R dQ/dt$ across it, and the differential equation governing the charge on the capacitor is then $L d^2Q/dt^2 + R dQ/dt + Q/C = 0$.

11.6.2 This is damped oscillatory motion, the condition for critical damping being $R^2 = 4L/C$. In fact, it is not necessary actually to have a physical resistor in the circuit. Even if the capacitor and inductor were connected by superconducting wires of zero resistance, while the charge in the circuit is slopping around between the capacitor and the inductor, it will be radiating electromagnetic energy into space and hence losing energy. The effect is just as if a resistance were in the circuit. If a battery of EMF E were in the circuit, the differential equation for Q would be $L d^2Q/dt^2 + R dQ/dt + Q/C = E$.

11.6.3 This is not quite an equation of the form 11.4.1, and I shan't spend time on it here. However, if we are interested in the current as a function of time, we just differentiate equation 11.6.3 with respect to time:

$L d^2I/dt^2 + R dI/dt + I = 0$. 11.6.4 The initial conditions are $I_0 = 0$ and $(dQ/dt)_0 = +E/L$, so the reader can easily write down all the solutions by comparison with section 11.5.

1 CHAPTER 12 FORCED OSCILLATIONS 12.1 More on Differential Equations In Section 11.4 we argued that the most general solution of the differential equation $ay'' + by' + cy = 0$ 11.4.1 is of the form $y = f_1(x) + Bg(x)$. 11.4.2 In this chapter we shall be looking at equations of the form $ay'' + by' + cy = h(x)$. 12.1.1 If you look back at the arguments that led to the conclusion that equation 11.4.2 is the most general solution of equation 11.4.1, you will

be able to conclude that 11.4.2 is still a solution of equation 11.4.1, but it is not the only solution. There is another function that is a solution, so that the most general solution to equation 12.1.1 is of the form $y = A f(x) + B g(x) + H(x)$. 12.1.2 The solution $H(x)$ is called the particular integral, while the part $A f(x) + B g(x)$ is the complementary function. I shall be dealing in this chapter mainly with the particular integral, though we shall not entirely forget the complementary function. This is a book on classical mechanics rather than on differential equations, so I am not going into how to obtain the particular integral $H(x)$ for a given $h(x)$. There are several ways of doing it; for those who know what they are and are in practice with them, Laplace transforms are among the more attractive methods. Some readers will already know how to do it. They will doubtless want to go back to equation 11.6.3 in the previous chapter and try their hand at finding the particular integral for that. Those who do not may be happy and content to take my word for the particular integral in the sections that follow, or perhaps at least to differentiate it to verify that it is indeed a solution. 12.2 Forced Oscillatory Motion. We are thinking of a mass m attached to a spring of force constant k and subject to a damping force $b \dot{x}$, but also subject to a periodic sinusoidal force $\cos \omega t$. The equation of motion is $m \ddot{x} + b \dot{x} + kx = P \cos \omega t$, 12.2.1

2 or, if we divide by m : $\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{P}{m} \cos \omega t$. 12.2.2 Here $\gamma = b/m$, $\omega_0 = k/m$, $\omega = 2\pi/\tau$ is the forcing angular frequency and ω_0 is the natural frequency of mass and spring in the absence of damping. One part of the general solution of equation 12.2.2 is the complementary function, which we have dealt with at length in Chapter 11. In this section I shall be interested in the particular integral. I shall not derive it here (those who are familiar with differential equations will be able to do so), but you should at least verify by differentiation and substitution that the following is a solution, and it is indeed the particular integral: $x = \frac{P}{m} \frac{1}{\omega_0^2 - \omega^2 + \gamma^2} [\omega_0^2 \cos \omega t + \gamma \omega \sin \omega t]$ 12.2.3 This can also be written $\frac{P}{m} \frac{\cos(\omega t - \phi)}{\sqrt{\omega_0^4 - 2\gamma^2 \omega^2 + \gamma^4}}$, 12.2.4 where $\cos \phi = \frac{\omega_0^2}{\sqrt{\omega_0^4 - 2\gamma^2 \omega^2 + \gamma^4}}$, since $\tan \phi = \frac{\gamma \omega}{\omega_0^2}$. 12.2.5 The response frequency is the same as the forcing frequency, but there is a phase lag between x and F . Figure XII. 1 shows ϕ as a function of $f = \omega/\omega_0$ for several different values of γ/ω_0 . The particular integral can also be written $x = \frac{2P}{m} \frac{\cos(\omega t - \phi)}{\omega_0^2 - \omega^2 + \gamma^2}$, 12.2.6 where the displacement amplitude varies with forcing frequency ω as 12.2.7 If we now introduce dimensionless quantities $f = \omega/\omega_0$, $\gamma = \gamma/\omega_0$, 12.2.8

0 =

We have now found the phase lag and the displacement and velocity amplitudes as a function of forcing frequency, but I must now try the reader's patience one step further for the most important part of the analysis, which really must not be skipped. Damping of oscillatory motion implies that some of the mechanical energy (which, in an undamped system, alternates between kinetic and potential energy) is lost - or, rather, that it is dissipated as heat. This happens if the damping is caused by the oscillator being immersed in a viscous fluid, or if it is caused by the repeated expansion and compression of a spring, or, in an electric circuit, by the dissipation of heat in the resistive part of the circuit. We aim now to find the rate at which the mechanical energy is dissipated as heat. We return to the equation of motion: $m\ddot{x} + b\dot{x} + kx = P \cos \omega t$, 12.2.1 and multiply each side by $\dot{x}$: $m\dot{x}\ddot{x} + b\dot{x}^2 + kx\dot{x} = \dot{x}P \cos \omega t$. 12.2.18 Introduce the total mechanical energy: $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ 12.2.19 $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$. The instantaneous rate of change of E is $\dot{E} = m\dot{x}\ddot{x} + kx\dot{x}$, while the instantaneous rate at which F does work is $\dot{W} = \dot{x}P \cos \omega t$. The difference (see equation 12.2.18), $b\dot{x}^2$, is therefore the rate at which work is being dissipated as heat, which, of course, is zero if $b = 0$. The average of $b\dot{x}^2$ over a complete period is $\overline{b\dot{x}^2} = \frac{b}{2} \overline{\dot{x}^2} = \frac{b}{2} (\omega^2 A^2) \overline{\sin^2(\omega t - \phi)}$, 12.2.20 where the bars denote the average value over a period. But $\sin^2(\omega t - \phi) = \frac{1}{2} [1 + \cos(2\omega t - 2\phi)]$, so the average rate at which work is being dissipated as heat, for which I shall use the symbol $\dot{Q}$, is $\dot{Q} = \frac{b}{2} \omega^2 A^2$. The reader should check that the right hand side has the dimensions of rate of dissipation of energy and hence the SI unit of watts. In dimensionless units, in which $\dot{Q}^* = \dot{Q} / (b^2 \omega^2 / 20)$ this can be written

$\dot{Q}^* = \frac{1}{2} (1 - \cos 2\phi)$ 12.2.22 $\dot{Q}^* = \frac{1}{2} (1 - \cos 2\phi)$ This is illustrated in figure XII.4. The reader can easily prove that the rate at which work is dissipated as heat is greatest when the forcing frequency is equal to ω_0 . 25 20 16 10 6 0 0 FIGURE XII.4 $F=0.2$ 06 1 1.5 Forcing frequency Summary. Phase lag: $\phi = \arctan(b\omega / (k - m\omega^2))$ (D O Displacement amplitude: $A = \frac{P}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$ 12.2.5 12.2.7

Velocity amplitude: $\dot{x} = A\omega \sin(\omega t - \phi)$ Rate of dissipation of work as heat: In terms of dimensionless variables, Phase lag: $\phi = \arctan(b\omega / (k - m\omega^2))$ 12.2.15 12.2.21 12.2.9 Displacement amplitude: $A = \frac{P}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$ 12.2.10 Velocity amplitude: $\dot{x} = A\omega \sin(\omega t - \phi)$ 12.2.16 Rate of dissipation of work as heat: $\dot{Q} = \frac{b}{2} \omega^2 A^2$ 12.2.22

9 12.3 Electrical Analogue Suppose that an alternating potential difference E

$\sin \cot$ is applied across an LCR circuit. We refer to equation 11.6.3, and we see that the equation that governs the charge on the capacitor is $LO + RO + Q = \sin \cot$. 12.3.1 C We can differentiate both sides with respect to time, and divide by L , and hence see that the current is given by $\cos \cot$. $L LC L$ We can compare this directly with equation 12.2.2, so that we have 12.3.2 $R^2 - 1/\cos$ 12.3.3 $\diamond = 7'$ $\cos$ Then, by comparison with equation 12.2.5, we see that ω will lag behind E by ϕ , where $\phi = \tan^{-1}(R/\omega L)$. 12.3.4 $\cos^2 Z \cos LC \cos$ This is just what we obtain from the more familiar complex number approach to alternating current circuits.

CHAPTER 13 LAGRANGIAN MECHANICS 13.1 Introduction The usual way of using newtonian mechanics to solve a problem in dynamics is first of all to draw a large, clear diagram of the system, using a ruler and a compass. Then mark in the forces on the various parts of the system with red arrows and the accelerations of the various parts with green arrows. Then apply the equation $F = ma$ in two different directions if it is a two-dimensional problem or in three directions if it is a three-dimensional problem, or $\tau = I \alpha$ if torques are involved. More correctly, if a mass or a moment of inertia is not constant, the equations are $F = \dot{p}$ and $\tau = \dot{L}$. In any case, we arrive at one or more equations of motion, which are differential equations which we integrate with respect to space or time to find the desired solution. Most of us will have done many, many problems of that sort. Sometimes it is not all that easy to find the equations of motion as described above. There is an alternative approach known as lagrangian mechanics which enables us to find the equations of motion when the newtonian method is proving difficult. In lagrangian mechanics we start, as usual, by drawing a large, clear diagram of the system, using a ruler and a compass. But, rather than drawing the forces and accelerations with red and green arrows, we draw the velocity vectors (including angular velocities) with blue arrows, and, from these we write down the kinetic energy of the system. If the forces are conservative forces (gravity, springs and stretched strings), we write down also the potential energy. That done, the next step is to write down the lagrangian equations of motion for each coordinate. These equations involve the kinetic and potential energies, and are a little bit more involved than $F = ma$, though they do arrive at the same results. I shall derive the lagrangian equations of motion, and while I am doing so, you will think that the going is very heavy, and you will be discouraged. At the end of the derivation you will see that the lagrangian

equations of motion are indeed rather more involved than $F = ma$, and you will begin to despair - but do not do so! In a very short time after that you will be able to solve difficult problems in mechanics that you would not be able to start using the familiar newtonian methods, and the speed at which you do so will be limited solely by the speed at which you can write. Indeed, you scarcely have to stop and think. You know straight away what you have to do. Draw the diagram. Mark the velocity vectors. Write down expressions for the kinetic and potential energies, and apply the lagrangian equations. It is automatic, fast, and enjoyable. Incidentally, when Lagrange first published his great work *La mbchanique analytique* (the modern French spelling would be *mbcanique*), he pointed out with some pride in his introduction that there were no drawings or diagrams in the book - because all of mechanics could be done analytically - i.e. with algebra and calculus. Not all of us, however, are as gifted as Lagrange, and we cannot omit the first and very important step

of drawing a large and clear diagram with ruler and compass and marking all the velocity vectors.

13.2 Generalized Coordinates and Generalized Forces

In two-dimensions the positions of a point can be specified either by its rectangular coordinates (x, y) or by its polar coordinates. There are other possibilities such as confocal conical coordinates that might be less familiar. In three dimensions there are the options of rectangular coordinates (x, y, z) , or cylindrical coordinates (ρ, ϕ, z) or spherical coordinates (r, θ, ϕ) - or again there may be others that may be of use for specialized purposes (inclined coordinates in crystallography, for example, come to mind). The state of a molecule might be described by a number of parameters, such as the bond lengths and the angles between the bonds, and these may be varying periodically with time as the molecule vibrates and twists, and these bond lengths and bond angles constitute a set of coordinates which describe the molecule. We are not going to think about any particular sort of coordinate system or set of coordinates. Rather, we are going to think about generalized coordinates, which may be lengths or angles or various combinations of them. We shall call these coordinates $(q_1, q_2, q_3, \dots)$. If we are thinking of a single particle in three-dimensional space, there will be three of them, which could be rectangular, or cylindrical, or spherical. If there were N particles, we would need $3N$ coordinates to describe the system - unless there were some constraints on the system. With each generalized coordinate q_i is associated a

generalized force P_i , which is defined as follows. If the work required to increase the coordinate q_i by δq_i is $P_i \delta q_i$, then P_i is the generalized force associated with the coordinate q_i . It will be noted that a generalized force need not always be dimensionally equivalent to a force. For example, if a generalized coordinate is an angle, the corresponding generalized force will be a torque. One of the things that we shall be wanting to do is to identify the generalized force associated with a given generalized coordinate. 13.3

Holonomic constraints The complete description of a system of N unconstrained particles requires $3N$ coordinates. You can think of the state of the system at any time as being represented by a single point in $3N$ -dimensional space. If the system consists of molecules in a gas, or a cluster of stars, or a swarm of bees, the coordinates will be continually changing, and the point that describes the system will be moving, perhaps completely unconstrained, in its $3N$ -dimensional space.

However, in many systems, the particles may not be free to wander anywhere at will; they may be subject to various constraints. A constraint that can be described by an equation relating the coordinates (and perhaps also the time) is called a holonomic constraint, and the equation that describes the constraint is a holonomic equation. If a system of N particles is subject to k holonomic constraints, the point in $3N$ -dimensional space that describes the system at any time is not free to move anywhere in $3N$ -dimensional space, but it is constrained to move over a surface of dimension $3N - k$. In effect only $3N - k$ coordinates are needed to describe the system, given that the coordinates are connected by k holonomic equations. Incidentally, I looked up the word "holonomic" in The Oxford English Dictionary and it said that the word was from the Greek holos , meaning "whole" or "entire" and nomos , meaning "law". It also said "applied to a constrained system in which the equations defining the constraints are integrable or already free of differentials, so that each equation effectively reduces the number of coordinates by one; also applied to the constraints themselves." As an example, consider a bar of wet soap slithering around in a hemispherical basin of radius a . You can describe its position in the basin by means of the usual two spherical angles (θ, ϕ) ; the motion is otherwise constrained by its remaining in contact with the basin; that is to say it is subject to the holonomic constraint $r = a$. Thus instead of needing three coordinates to describe the position of a totally unconstrained particle, we need only two

Two unconstrained particles would require six coordinates to specify their positions but this system is subject to four holonomic constraints. The holonomic equations $z = 0$

given by $p_j = F_i \cdot r_i$. 13.4.4 i Now $F = m$): , so that 13.4.5

13.4.9

substituting these in equation 13.4.8 we obtain d OT OT P. 13.4.12 This is

one form of Lagrange's equation of motion, and it often helps us to answer the question posed in the past sentence of section 13.2 - namely to determine the generalized force associated with a given generalized coordinate. If the various forces in a particular problem are conservative (gravity, springs and stretched strings, including valence bonds in a molecule) then the generalized force can be obtained by the negative of the gradient of a potential energy function - i.e. $Q_i = -\partial V / \partial q_i$. In that case, Lagrange's equation takes the form $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = Q_i$. 13.4.13 In my experience this is the most useful and most often encountered version of Lagrange's equation. The quantity $L = T - V$ is known as the lagrangian for the system, and Lagrange's equation can then be written $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$. 13.4.14 It is my understanding that this last version is of deep theoretical significance, although I don't recall my ever having used it in solving any actual problem. Equation 13.4.13 is my favorite, provided the forces are conservative. (For conservative forces, see Chapter 9, though in practice it means gravity, springs and strings.)

In each of equations 13.4.12, 13 and 14 one of the q s has a dot over it. You can see which one it is by thinking about the dimensions of the various terms. Dot has dimension T^{-1} . So, we have now derived Lagrange's equation of motion. It was a hard struggle, and in the end we obtained three versions of an equation which at present look quite useless. But from this point, things become easier and we rapidly see how to use the equations and find that they are indeed very useful. 13.5 Acceleration Components In section 3.4 of chapter 3 of the Celestial Mechanics "book", I derived the radial and transverse components of velocity and acceleration in two-dimensional coordinates. The radial and transverse velocity components are fairly obvious and scarcely need derivation; they are just $\dot{r}$ and $r\dot{\theta}$ (p 4). For the acceleration components I reproduce here an extract from that chapter: "The radial and transverse components of acceleration are therefore $\ddot{r} - r\dot{\theta}^2$ and $r\ddot{\theta} + 2\dot{r}\dot{\theta}$ respectively." I also derived the radial, meridional and azimuthal components of velocity and acceleration in three-dimensional spherical coordinates. Again the velocity components are rather obvious; they are $\dot{r}$, $r\dot{\theta}$ and $r\sin\theta\dot{\phi}$ (p 4), while for the acceleration components I reproduce here the relevant extract from that chapter. "On gathering together the coefficients of $\ddot{r}$, $\ddot{\theta}$, $\ddot{\phi}$, we find that the components of acceleration are: Radial: $\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2$; Meridional: $\ddot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\theta}\dot{\phi}^2\sin\theta\cos\theta$; Azimuthal: $2\dot{r}\dot{\theta}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\sin\theta\ddot{\phi}$." You might like to look back at these derivations now. However, I am now

going to derive them by a different method, using Lagrange's equation of motion. You can decide for yourself which you prefer. We'll start in two dimensions. Let R and S be the radial and transverse components of a force acting on a particle. ("Radial" means in the direction of increasing p ; "transverse" means in the direction of increasing ϕ .) If the radial coordinate were to increase by

the work done by the force would be just $R \delta p$. Thus the generalized force associated with the coordinate p is just $P_p = R$. If the azimuthal angle were to increase by $\delta\phi$, the work done by the force would be $S \delta\phi$. Thus the generalized force associated with the coordinate ϕ is $P_\phi = S$. Now we don't have to think about how to start; in Lagrangian mechanics, the first line is always " $T = \dots$ ", and I hope you'll agree that $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$. 13.5.1 If you now apply equation 13.4.12 in turn to the coordinates p and ϕ , you obtain $P_r = m(\ddot{r} - r\dot{\phi}^2)$ and $P_\phi = m(r^2\ddot{\phi} + 2r\dot{r}\dot{\phi})$, 13.5.2a,b and so $R = m(\ddot{r} - r\dot{\phi}^2)$ and $S = m(r^2\ddot{\phi} + 2r\dot{r}\dot{\phi})$. 13.5.3a,b Therefore the radial and transverse components of the acceleration are $(\ddot{r} - r\dot{\phi}^2)$ and $(r\ddot{\phi} + 2\dot{r}\dot{\phi})$ respectively. We can do exactly the same thing to find the acceleration components in three-dimensional spherical coordinates. Let R , S and F be the radial, meridional and azimuthal (i.e. in direction of increasing r , θ and ϕ) components of a force on a particle. If r increases by δr , the work done is $R \delta r$. If θ increases by $\delta\theta$, the work done is $S \delta\theta$. If ϕ increases by $\delta\phi$, the work done is $F \delta\phi$. Therefore $P_r = R$, $P_\theta = S$ and $P_\phi = F$. Start: $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2)$. If you now apply equation 13.4.12 in turn to the coordinates r , θ and ϕ , you obtain and 13.5.4 Therefore $P_r = m(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2)$, $P_\theta = m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\sin\theta\cos\theta\dot{\phi}^2)$, $P_\phi = m(r^2\sin^2\theta\ddot{\phi} + 2r\dot{r}\sin^2\theta\dot{\phi} + 2r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi})$. $R = m(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2)$, $S = m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\sin\theta\cos\theta\dot{\phi}^2)$ 13.5.5 13.5.6 13.5.7 13.5.8 13.5.9

and $F = m(r^2\sin^2\theta\ddot{\phi} + 2r\dot{r}\sin^2\theta\dot{\phi} + 2r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi})$. Thus the acceleration components are Radial: Meridional: Azimuthal: 13.5.10 $(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2)$, $(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\sin\theta\cos\theta\dot{\phi}^2)$, $(r^2\sin^2\theta\ddot{\phi} + 2r\dot{r}\sin^2\theta\dot{\phi} + 2r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi})$. Be sure to check the dimensions. Since dot has dimension T^{-1} , and these expressions must have the dimensions of acceleration, there must be an r and two dots in each term. 13.6 Slithering Soap in Conical Basin We imagine a slippery (no friction) bar of soap slithering around in a conical basin. An isolated bar of soap in intergalactic space would require three coordinates to specify its position at any time, but, if it is subject to the holonomic constraint that it is to be in contact at all times

with a conical basin, its position at any time can be specified with just two coordinates. I shall, first of all, analyse the problem with a newtonian approach, and then, for comparison, I shall analyse it using lagrangian methods. Either way, we start with a large diagram. In the newtonian approach we mark in the forces in red and the accelerations in green. See figure XIII.2. The semi vertical angle of the cone is c . R $r \sin \theta$ mg FIGURE XIII.2

10 The two coordinates that we need are r , the distance from the vertex, and the azimuthal angle ϕ , which I'll ask you to imagine, measured around the vertical axis from some arbitrary origin. The two forces are the weight mg and the normal reaction R of the basin on the soap. The accelerations are $\ddot{r}$ and the centripetal acceleration as the soap moves at angular speed $\dot{\phi}$ in a circle of radius $r \sin \theta$ is $r \sin \theta \dot{\phi}^2$. We can write the newtonian equation of motion in various directions: Horizontal: $R \cos \theta = m(r \sin \theta \dot{\phi}^2 - \ddot{r} \sin \theta)$ i.e. $R = m \tan \theta (r \dot{\phi}^2 - \ddot{r})$. 13.6.1 Vertical: $R \sin \theta - mg = m \ddot{r} \cos \theta$. 13.6.2 Perpendicular to surface: $R - mg \sin \theta = m r \sin \theta \cos \theta \ddot{\theta}$. 13.6.3 Parallel to surface: $g \cos \theta = r \sin^2 \theta \ddot{\phi} + 2 \dot{r} \dot{\phi} \sin \theta$. 13.6.4 Only two of these are independent, and we can choose to use whichever two we want to at our convenience. There are, however, three quantities that we may wish to determine, namely the two coordinates r and ϕ , and the normal reaction R . Thus we need another equation. We note that, since there are no azimuthal forces, the angular momentum per unit mass, which is $r^2 \sin^2 \theta \dot{\phi}$, is conserved, and therefore $r^2 \sin^2 \theta$ is constant and equal to its initial value, which I'll call $l^2 C^2$. That is, we start off at a distance l from the vertex with an initial angular speed g . Thus we have as our third independent equation $r^2 \sin^2 \theta \dot{\phi} = l^2 C^2$. 13.6.5 This last equation shows that $\dot{\phi} \rightarrow \infty$ as $r \rightarrow 0$. One possible type of motion is circular motion at constant height (put $\theta = 0$). equations 13.6.1 and 2 it is easily found that the condition for this is that $g \sin \theta = \tan \theta$

11 In other words, if the particle is projected initially horizontally ($\dot{\theta} = 0$) at $r = l$ and $\dot{\phi} = C$, it will describe a horizontal circle (for ever) if $l^2 C^2 = g \sin \theta \cos \theta$, say $l^2 C^2 = g \sin^2 c$. 13.6.7 If the initial speed is less than this, the particle will describe an elliptical orbit with a minimum $r < l$; if the initial speed is greater than this, the particle will describe an elliptical orbit with a maximum $r > l$. Now let's do the same problem in a lagrangian formulation. This time

we draw the same diagram, but we mark in the velocity components in blue. See figure XIII.3. We are dealing with conservative forces, so we are going to use equation 13.4.13, the most useful form of Lagrange's equation. $r \sin \theta$

FIGURE XIII.3 We need not spend time wondering what to do next. The first and second things we always have to do are to find the kinetic energy T and the potential energy V , in order that we can use equation 13.4.13. $\frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2$

◆ $m(r + \sin \theta)$ 13.6.8

12 and $V = mgr \cos \theta + \text{constant}$. 13.6.9 Now go to equation 13.4.13, with $q_1 = r$, and work out all the derivatives, and you should get, when you apply the lagrangian equation to the coordinate r : $-\frac{1}{2} m r \sin^2 \theta = -mg \cos \theta$. 13.6.10

Now do the same thing with the coordinate θ . You see immediately that $\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} = 0$ and $\frac{\partial V}{\partial \theta} = 0$. Therefore $\frac{\partial T}{\partial \dot{\theta}}$ is constant and therefore $r^2 \dot{\theta}$ is constant. That is, $m r^2 \dot{\theta}$ is constant and so $r^2 \dot{\theta}$ is constant and equal to its initial value l^2 . Thus the second lagrangian equation is $r \ddot{\theta} = l \dot{\theta}^3$. 13.6.11

Since the lagrangian is independent of θ , θ is called, in this connection, an "ignorable coordinate" - and the momentum associated with it, namely $m r^2 \dot{\theta}$, is constant. Now it is true that we arrived at both of these equations also by the newtonian method, and you may not feel we have gained much. But this is a simple, introductory example, and we shall soon appreciate the power of the lagrangian method. Having got these two equations, whether by newtonian or lagrangian methods, let's explore them further. For example, let's eliminate θ between them and hence get a single equation in r : $\frac{1}{4} m \dot{r}^2 \sin^2 \theta = m g r \cos \theta$. 13.6.12

We know enough by now (see Chapter 6) to write $\dot{\theta}$ as v , where $v = h$, and if we do let the constants $\frac{1}{4} m \dot{r}^2 \sin^2 \theta$ and $m g r \cos \theta$ equal A and B respectively, equation 13.6.12 becomes $\frac{dv}{dr} = \frac{A}{v} - \frac{B}{r^3}$. 13.6.13

(It may just be useful to note that the dimensions of A and B are $L^4 T^{-2}$ and $L T^{-2}$ respectively. This will enable us to keep track of dimensional analysis as we go.)

13 If we start the soap moving horizontally ($\dot{\theta} = 0$) when $r = l$, this integrates, with these initial conditions, to $\frac{1}{2} m \dot{r}^2 + 2B(l - r)$. 13.6.14

Again, so that we can see what we are doing, let $\frac{1}{2} m \dot{r}^2 + 2B(l - r) = C$ and equation 13.6.14 becomes (note that $[C]: L^2 T^{-2}$), $\frac{1}{2} m \dot{r}^2 = C - 2B(r - l)$. 13.6.15

This gives $\dot{r}$ as a function of r . The particle reaches its maximum or minimum height when $\dot{r} = 0$; that is where $2B(r - l) - C = 0$. 13.6.16

We already know one root of this cubic equation, namely $r = l$, because our initial condition was that $\dot{r} = 0$.

= 0 at $r = l$, and, with A , B and C all positive, from the theory of equations, there are no more than two positive real roots. Example. With initial conditions $r = 1$ and $b = f/2c$, we obtain $A = 4/3 g \cos \theta$, $B = g \cos \theta$ and $C = 6 g l \cos \theta$. With $r' = r/l$, the equation becomes $3r'^2 + 2 = 0$. 13.6.17 The two positive real roots are $r' = 1$ (which we already know) and $1 + x/2$. (The negative real root is $1 - x/2$). Exercise. Show that if the initial condition is $b = f/c$ the low point is reached at $r = 0.42151$. Exercise. Show that if the initial condition is $b = c/l$ the low or high point is reached at the positive root of the equation $2r^2 - c^2 r - c^2 = 0$. Show that this agrees with the two previous examples. What is the root if $c = 17$ What if c is very small - say 0.017 What if it is zero?

14 13.7 Slithering Soap in Hemispherical Basin Suppose that the basin is of radius a and the soap is subject to the holonomic constraint $r = a$ - i.e. that it remains in contact with the basin at all times. Note also that this is just the same constraint of a pendulum free to swing in three-dimensional space except that it is subject to the holonomic constraint that the string be taut at all times. Thus any conclusions that we reach about our soap will also be valid for a pendulum. We'll start with the newtonian approach, and I'll draw in red the two forces on the soap, namely its weight and the normal reaction of the basin on the soap. Figure XIII.4 FIGURE XIII.4 We'll make use of the expressions for the radial, meridional and azimuthal accelerations from section 13.5 and we'll write down the equations of motion in these directions: Radial: $mg \cos \theta - R$ Meridional: $-rag \sin \theta$ Azimuthal: $0 : m(\ddot{\theta} - r)$: $-r \sin \theta \ddot{\phi}$), 13.7.1 $m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) - r \sin \theta \cos \theta \ddot{\phi}$), 13.7.2 $m(2b(\sin \theta + 2r) \cos \theta + r \sin \theta \ddot{\phi})$. 13.7.3 We also have the constraint that $r = a$ and hence that $\dot{r} = \ddot{r} = 0$, equations become after which these $mg \cos \theta - R = -ma(\ddot{\theta} + \sin^2 \theta \ddot{\phi})$, 13.7.4

15 $-g \sin \theta = a(\ddot{\theta} - \sin \theta \cos \theta \ddot{\phi})$, 13.7.5 $0 = \sin \theta \ddot{\phi} + 2b \cos \theta$. 13.7.6 These, then, are the newtonian equations of motion. If you still prefer the newtonian method to the lagrangian method, and you wish to integrate these and find expressions for θ , $\dot{\theta}$ and R separately, by all means go ahead and do so - but I'm now going to try the lagrangian approach. Although Lagrange himself would not have drawn a diagram, we shall not omit that step - but instead of marking in the forces, we'll mark in the velocity components, and then we'll immediately write down expressions for the kinetic and potential energies.

Indeed the first line of a lagrangian calculation is always "T = ...". In O(b
 FIGURE XIII.5 T : $3ma^2(\dot{\theta})^2 + \sin 2\theta$). 13.7.7 $V = -mga \cos \theta + \text{constant}$.
 13.7.8 Now apply equation 13.4.13 in turn to the coordinates θ and ϕ . 0: $a\dot{\theta} \sin \theta \cos \theta - g \sin \theta$. 13.7.9

16 T V 4: As for the conical basin, we see that $\ddot{\theta}$ and $\ddot{\phi}$ are both zero (θ is an "ignorable T coordinate") and therefore is constant and equal to its initial value. If the initial values of b and θ are b_0 and θ_0 respectively, then $\sin^2 \theta_0 (b_0 = \sin^2 \theta_0 a^2 \dot{\phi}^2)$. This is merely stating that angular momentum is conserved. We can easily eliminate b from equations 13.4.9 and 10 to obtain 13.7.10 $i = k \cot \theta \csc^2 \theta g \sin \theta$ 13.7.11 where $k = \sin^4 \theta_0 a^2 \dot{\phi}_0^2 / 2$. 13.7.12 $\frac{d\theta}{dt}$ Write i as in the usual way and integrate to obtain the first space integral: $\frac{d\theta}{dt}^2 = 2g (\cos \theta_0 - \cos \theta) - k(\csc^2 \theta_0 - \csc^2 \theta)$. 13.7.13 a The upper and lower bounds for θ occur when $\dot{\theta} = 0$. Example. Suppose that the initial value of θ is $\theta_0 = 45^\circ$ and that we start by pushing the soap horizontally ($\dot{\theta} = 0$) at an initial angular speed $\dot{\phi}_0 = 3 \text{ rad/s}$, so that $k = 2.25 \text{ rad}^2/\text{s}^2$. Suppose that the radius of the basin is $a = 1.96 \text{ m}$ and that $g = 9.8 \text{ m/s}^2$. You can then solve equation for $\dot{\theta} = 0$. One solution, of course, is $\theta = \theta_0$. We could find the other solution by Newton-Raphson iteration, or by putting $\csc^2 \theta_0 = 1/(1 - \cos^2 \theta_0)$ and solving it as a cubic equation in $\cos \theta$. Alternatively, try this: $a k \frac{2g}{a} = l$, $(f/\sin \theta_0)^2 - 1 \cos \theta_0 = m$, $= n$, $\cos \theta_0 = x$, $\cos \theta = c$, $2g$ so that $\csc^2 \theta_0 = 1/(1 - x^2)$ and $\csc^2 \theta = 1/(1 - c^2)$. The equation then becomes (work through the algebra - don't just take my word for it) $(x - c)[(c^2 - 1)x^2 - nx + 1 - c(m + c)] = 0$.

17 One solution is obviously θ becomes $-0.5x$: In our example the quadratic part of the equation is $-0.225x^2 + 0.340900974x = 0$. The only real solution for θ is 50° $\approx 53^\circ$. 13.8 i. More Examples ml FIGURE XIII.6 5+) m^2 The upper pulley is fixed in position. Both pulleys rotate freely without friction about their axles. Both pulleys are "light" in the sense that their rotational inertias are small and their rotation contributes negligibly to the kinetic energy of the system. The rims of the pulleys are rough, and the ropes do not slip on the pulleys. The gravitational acceleration is g . The mass M accelerates upwards at a rate $\ddot{y}$ with respect to the upper, fixed, pulley, and the smaller pulley accelerates downwards at the same rate. The mass m accelerates upwards at a rate $\ddot{y}$ with respect to the small pulley, and consequently its acceleration in laboratory space is $2\ddot{y}$. The acceleration of the mass m^2 is therefore $+\ddot{y}$ in laboratory space. The object is to find $\ddot{y}$ in terms of g . The kinetic energy

is $T = M\dot{x} + m(\dot{x} - a\dot{\theta}) + m(\dot{x} + a\dot{\theta})$. 13.8.1 The potential energy is $V = g[Mx - m(x - y) - m^2(x + y)] + \text{constant}$. 13.8.2 Apply Lagrange's equation (13.4.13) in turn to the coordinates x and y :

13.8.3 $y: -m(\ddot{x} - a\ddot{\theta}) + m(\ddot{x} + a\ddot{\theta}) = -g(m - m)$. 13.8.4 These two equations can be solved at one's leisure for $\ddot{x}$ and $\ddot{\theta}$. A hoop of mass M and radius a rolls without slipping on a horizontal plane. A bead of mass m slides smoothly around the rim of the hoop. Describe the motion. FIGURE XIII.7 I have marked in the several velocity vectors. The hoop is rolling at angular speed $\dot{\theta}$. Consequently the linear speed of the centre of mass of the hoop is $a\dot{\theta}$, and the bead also shares this velocity. In addition, the bead is sliding relative to the hoop at an angular speed $\dot{\phi}$ and consequently has a component to its velocity of $a\dot{\phi}$ tangential to the hoop. We are now ready to start. The kinetic energy of the hoop is the sum of its translational and rotational kinetic energies: $\frac{1}{2}M\dot{x}^2 + \frac{1}{2}Ma^2\dot{\theta}^2$. The kinetic energy of the bead is

13.8.5 The potential energy is $V = \text{constant} - mga\cos\theta$. 13.8.6 The lagrangian equation in θ becomes $a(\ddot{\theta} - \cos\theta) + g\sin\theta = 0$. 13.8.7 The lagrangian equation in ϕ becomes $(2M + m)\ddot{\phi} = m(\ddot{\theta} - \sin\theta)$. 13.8.8 (Some algebra was needed to arrive at these. If you need help, let me know.) These, then, are two differential equations in the two variables. The lagrangian part of the analysis is over; we now have to see if we can do anything with these equations. It is easy to eliminate $\ddot{\phi}$ and hence get a single differential equation in θ : $(2M + m\sin^2\theta)a\ddot{\theta} + m\sin\theta\cos\theta\dot{\theta}^2 + (2M + m)g\sin\theta = 0$. 13.8.9 If you are good at differential equations, you might be able to do something with this, and get θ as a function of the time. In the meantime, I think I can get the "first space integral" (see Chapter 6) - i.e. θ as a function of $\dot{\theta}$. Thus, the total energy is constant: $\frac{1}{2}M\dot{x}^2 + \frac{1}{2}ma^2(\dot{\theta} + \dot{\phi})^2 - mga\cos\theta = E$. 13.8.10 Here I am measuring the potential energy from the centre of the circle. Also, if we assume that the initial condition is that at time $t = 0$ the kinetic energy was zero and $\theta = c$, then $E = -mga\cos c$. Equation 13.8.8 can easily be integrated once with respect to time, since $\dot{\theta}\cos\theta - \dot{\theta}\sin\theta = d(\dot{\theta}\cos\theta)$ as would have been apparent during the derivation of $\ddot{\theta}$ equation 13.8.8. With the condition that the kinetic energy was initially zero, integration of equation 13.8.8 gives

$\frac{1}{2}(2M + m)b^2 = m\dot{\theta}^2 \cos\theta$. 13.8.11 Now we can easily eliminate b between

equations 13.8.10 and 11, to obtain a single equation relating θ and $\dot{\theta}$: $b^2(1 + c \sin^2 \theta) - d \cos \theta - 1 = 0$, 13.8.12 $Mma + m m g a$ where $b = (2M+m)\omega$ $c = d/g$ secc. 13.8.13a,b,c $2M/g$ iii. FIGURE XIII.8 As in example ii, we have a hoop of radius a and mass M , and a ring of mass m which can slide freely and without friction around the hoop. This time, however, the hoop is not rolling along the table, but is spinning about a vertical axis at an angular speed b . The ring has a velocity component $a\dot{\theta}$ because it is sliding around the hoop, and a component $a \sin \theta \dot{\phi}$ because the hoop is spinning. The resultant speed is the orthogonal sum of

21 these. the ring and the rotational kinetic energy of the hoop: $\frac{1}{2} T = \frac{1}{2} m a^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) + \frac{1}{2} (M a^2) b^2$. 13.8.14 If we refer potential energy to the centre of the hoop: $V = m g a \cos \theta$. 13.8.15 The lagrangian equations with respect to the two variables are $0: a(\ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2) - g \sin \theta = 0$. 13.8.16 $\frac{1}{2} \dot{\phi}^2 + m \sin \theta \dot{\theta} + M \dot{\phi} = \text{constant}$. 13.8.17 The constant is equal to whatever the initial value of the left hand side was. E.g., maybe the initial values of θ and $\dot{\phi}$ were θ_0 and $\dot{\phi}_0$. This finishes the lagrangian part of the analysis. The rest is up to you. For example, it would be easy to eliminate $\dot{\phi}$ between these two equations to obtain a differential equation between θ and the time. If you then write $\dot{\theta} = d\theta/dt$ as $d\theta/d\phi$ in the usual way, I think it wouldn't be too difficult to obtain the first space integral and hence get $\dot{\theta}$ as a function of θ . I haven't tried it, but I'm sure it'll work. The kinetic energy of the system is the sum of the translational kinetic energy of iv. FIGURE XIII.9

22 Figure XIII. 10 shows a pendulum. The mass at the end is m . It is at the end not of the usual inflexible string, but of an elastic spring obeying Hooke's law, of force constant k . The spring is sufficiently stiff at right angles to its length that it remains straight during the motion, and all the motion is restricted to a plane. The unstretched natural length of the spring is l , and, as shown, its extension is r . The spring itself is "light" in the sense that it does not contribute the the kinetic or potential energies. (You can give the spring a finite mass if you want to make the problem more difficult.) The kinetic and potential energies are $T = \frac{1}{2} m (\dot{h}^2 + l^2 \dot{\theta}^2)$ and $V = \text{constant} - m g (l+r) \cos \theta + \frac{1}{2} k r^2$. 13.8.18 12 and $V = \text{constant} - m g (l+r) \cos \theta + \frac{1}{2} k r^2$. 13.8.19 Apply Lagrange's equation in turn to r and to θ and see where it leads you. v. Another example suitable for lagrangian methods is given as problem number 11 in Appendix A of these notes. Lagrangian methods are particularly applicable to vibrating systems, and

examples of these will be discussed in a later chapter. Since these chapters are being written in more or less random order as the spirit moves me, rather than in logical order, present indications are that it is likely to be chapter 17 - after the unlikely sequence of relativity and hydrostatics.

13.9 Hamilton's Variational Principle

Hamilton's variational principle in dynamics is slightly reminiscent of the principle of virtual work in statics, discussed in section 9.4 of Chapter 9. When using the principle of virtual work in statics we imagine starting from an equilibrium position, and then increasing one of the coordinates infinitesimally. We calculate the virtual work done and set it to zero. I am slightly reminded of this when discussing Hamilton's principle in dynamics. Imagine some mechanical system - some contraption including in its construction various wheels, jointed rods, springs, elastic strings, pendulums, inclined planes, hemispherical bowls, and ladders leaning against smooth vertical walls and smooth horizontal floors. It may require N generalized coordinates to describe its configuration at any time. Its configuration could be described by the position of a point in N -dimensional space. Or perhaps it is subject to k holonomic constraints - in which case the point that describes its configuration in N -dimensional space is not free to move

anywhere in that space, but is constrained to slither around on a surface of dimension $N-k$. The system is not static, but it is evolving. It is changing from some initial state at time t to some final state at time t_2 . The generalized coordinates that describe it are changing with time - and the point in N -space is slithering round on its surface of dimension $N-k$. One can imagine that at any instant of time one can calculate its kinetic energy T and its potential energy V , and hence its lagrangian $L = T - V$. You can multiply L at some moment by a small time interval Δt and then add up all of these products between t and t_2 to form the integral. This quantity - of dimension ML^2T^{-1} and SI unit $J \cdot s$ - is sometimes called the "action". There are many different ways in which we can imagine the system to evolve from its initial state to its final state - and there are many different routes that we can imagine might be taken by our point in N -space as it moves from its initial position to its final position, as long as it moves over its surface of dimension $N-k$. But, although we can imagine many such routes, the manner in which the system will actually evolve, and the route that the point will actually take is determined by Hamilton's principle; and the route, according to this principle,

is such that the integral $\int_{t_1}^{t_2} L dt$ is a minimum, or a maximum, or an inflection point, when compared with other imaginable routes. Stated otherwise, let us suppose that we calculate $\int_{t_1}^{t_2} L dt$ over the actual route taken and then calculate the variation in $\int_{t_1}^{t_2} L dt$ if the system were to move over a slightly different adjacent path. Then (and here is the analogy with the principle of virtual work in a statics problem) this variation from what $\int_{t_1}^{t_2} L dt$ would have been over the actual route is zero. And this is Hamilton's variational principle. The next questions will surely be: Can I use this principle for solving problems in mechanics? Can I prove this bald assertion? Let me try to use the principle to solve two simple and familiar problems, and then move on to a more general problem.

24 The first problem will be this. Imagine that we have a particle that can move in one dimension (i.e. one coordinate - for example its height y above a table - suffices to describe its position), and that when its coordinate is y its potential energy is $V = mgy$. 13.9.1 Its kinetic energy is, of course, $T = \frac{1}{2} m \dot{y}^2$. 13.9.2 We are going to use the variational principle to find the equation of motion - i.e we are going to find an expression for its acceleration. I imagine at the moment you have no idea what its acceleration could possibly be - but don't worry, for we know that the lagrangian is $L = \frac{1}{2} m \dot{y}^2 - mgy$, 13.9.3 and we'll make short of it with Hamilton's variational principle and soon find the acceleration. According to this principle, y must vary with t in such a manner that $\delta \int_{t_1}^{t_2} (\frac{1}{2} m \dot{y}^2 - mgy) dt = 0$. 13.9.4 Let us vary y by δy and $\dot{y}$ by $\delta \dot{y}$, and see how the integral varies. The integral is then $\int_{t_1}^{t_2} (\frac{1}{2} m \dot{y}^2 - mgy) dt$ which I'll call I . Now $\delta I = \int_{t_1}^{t_2} (m \dot{y} \delta \dot{y} - mg \delta y) dt$. 13.9.5 which I'll call I . Now $\delta I = \int_{t_1}^{t_2} (m \dot{y} \delta \dot{y} - mg \delta y) dt$. 13.9.6. (If unconvinced of this, consider $\int_{t_1}^{t_2} \cos t dt = \sin t$ and $\int_{t_1}^{t_2} \sin t dt = -\cos t$.) By integration by parts: 13.9.10

25 The first term is zero because the variation is zero at the beginning and end points. In the second term, $d(\dot{y} \delta y) = \dot{y} \delta y + \ddot{y} \delta y$, and therefore $I = -m \int_{t_1}^{t_2} (\ddot{y} \delta y + g \delta y) dt$, and, for this to be zero, we must have $\ddot{y} + g = 0$. This is the equation of motion that we sought. would you? 13.9.11 13.9.12 13.9.13 You would never have guessed this, Now let's do another one-dimensional problem. Only one coordinate, x , describes the particle's position, and, when its coordinate is x we'll suppose that its potential energy is $V = \frac{1}{2} m \omega^2 x^2$. 1. 2. 2 1. 2 $V = \frac{1}{2} m \omega^2 x^2$: , and its kinetic energy is, of course, $T = \frac{1}{2} m \dot{x}^2$. The equation of

motion, or the way in which the acceleration varies with position, must be such as to satisfy $\int_1^2 \frac{d}{dt} \left(\frac{1}{2} m \dot{x}^2 - m \phi(x) \right) dt = 0$. 13.9.14 If we vary ϕ by $\delta\phi$ and x by δx , the variation in the integral will be $m \int_1^2 \left(\dot{x} \delta \dot{x} - \delta \phi - m \dot{x} \delta x \right) dt = I_2 - I_1$, say. 13.9.15 By precisely the same argument as before, the first integral is found to be $-m \dot{x} \delta x + \int_1^2 \delta \phi dt - m \int_1^2 \dot{x} \delta x dt$, 13.9.16 Therefore $\frac{d}{dt} \left(\frac{1}{2} m \dot{x}^2 - m \phi(x) \right) = 0$ and, for this to be zero, we must have $\ddot{x} = -\phi'(x)$. 13.9.17 These two examples must have given the impression that we are doing something very difficult in order to derive something that is immediately obvious - but the examples were just intended to show the direction of a more general argument we are about to make.

26 This time, we'll consider a very general system, in which we write the lagrangian as a function of the (several) generalized coordinates and their time rates of change - i.e. $L = L(q_i, \dot{q}_i, t)$ - without specifying any particular form of the function - and we'll carry out the same sort of argument to derive a very general equation of motion. We have $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}$, 13.9.18 As before, $\frac{\partial L}{\partial q_i} = \frac{\partial L}{\partial q_i} + \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i}$, so that $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$. 13.9.20 Thus we arrive at the general equation of motion $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}$. 13.9.21 Thus we have derived Lagrange's equation of motion from Hamilton's variational principle, and this is indeed the way it is often derived. However, in the chapter, I derived Lagrange's equation quite independently, and hence I would regard this derivation not so much as a proof of Lagrange's equation but as a vindication of the correctness of Hamilton's variational principle.

27

CHAPTER 14 HAMILTONIAN MECHANICS 14.1 Introduction The hamiltonian equations of motion are of deep theoretical interest. Having established that, I am bound to say that I have not been able to think of a problem in classical mechanics that I can solve more easily by hamiltonian methods than by newtonian or lagrangian methods. That is not to say that real problems cannot be solved by hamiltonian methods. What I have been looking for is a problem which I can solve easily by hamiltonian methods but which is more difficult to solve by other methods. So far, I have not found one. Having said that, doubt not that hamiltonian mechanics is of deep theoretical significance. Having expressed that mild degree of cynicism, let it

be admitted that Hamilton theory - or more particularly its extension the Hamilton-Jacobi equations - does have applications in celestial mechanics, and of course hamiltonian operators play a major part in quantum mechanics, although it is doubtful whether Sir William would have recognized his authorship in that connection.

14.2 A Thermodynamics Analogy Readers may have noticed from time to time - particularly in Chapter 9- that I have perceived some connection between parts of classical mechanics and thermodynamics. I perceive such an analogy in developing hamiltonian dynamics. Those who are familiar with thermodynamics may also recognize the analogy. Those who are not can skip this section without seriously prejudicing their understanding of subsequent sections. Please do not misunderstand: The hamiltonian in mechanics is not at all the same thing as enthalpy in thermodynamics, even though we use the same symbol, H . Yet there are similarities in the way we can introduce these concepts. In thermodynamics we can describe the state of the system by its internal energy, defined in such a way that when heat is supplied to a system and the system does external work, the increase in internal energy of the system is equal to the heat supplied to the system minus the work done by the system: $dU = T dS - P dV$.

14.2.1 From this point of view we are saying that the internal energy is a function of the entropy and the volume: $U = U(S, V)$

14.2.2

so that from which we see that $dU = U + dV$, 14.2.3 $v \propto T$ and $-P \propto -/$ 14.2.4,5 $v \propto S$ However, it is sometimes convenient to change the basis of the description of the state of a system from S and V to S and P by defining a quantity called the enthalpy H defined by $H = U + PV$. 14.2.6 In that case, if the state of the system changes, then $dH = dU + PdV + VdP$ 14.2.7 $= TdS - PdV + PdV + VdP$. 14.2.8 I.e. $dH = TdS + VdP$. 14.2.9 Thus we see that, if heat is added to a system held at constant volume, the increase in the internal energy is equal to the heat added; whereas if heat is added to a system held at constant pressure, the increase in the enthalpy is equal to the heat added. From this point of view we are saying that enthalpy is a function of entropy and pressure: $H = H(S, P)$ 14.2.10 so that $dH = H dS + dP$, 14.2.11 $P \propto S$ from which we see that None of this has anything to do with hamiltonian dynamics, so let's move on. 14.2.12

14.3 Hamilton's Equations of Motion In classical mechanics we can describe the state of a system by its lagrangian:

◆: 14.3.2

(I am deliberately numbering this equation 14.3.2, to maintain an analogy between this section and section 14.2.) If the coordinates and the velocities change, the corresponding change in the lagrangian is 14.3.3 Definition.' The generalized momentum p_i associated with the generalized coordinate q_i is defined as 14.3.4 It follows from the lagrangian equation of motion $dL/d\dot{q}_i = p_i$ (equation 13.4.14) that $dL = \sum_i p_i dq_i + \dots$ 14.3.5 Oq Thus $dL = \sum_i p_i dq_i + \dots$ 14.3.1 i (I am deliberately numbering this equation 14.3.1, to maintain an analogy between this section and section 14.2.) However, it is sometimes convenient to change the basis of the description of the state of a system from q_i and $\dot{q}_i$ to q_i and p_i by defining a quantity called the hamiltonian H defined by Definition 14.3.6 In that case, if the state of the system changes, then $dH = \sum_i p_i dq_i + \dots$ 14.3.7 i i i i i i 14.3.8 I.e. $dH = \sum_i p_i dq_i + \dots$ 14.2.9 i i

We are regarding the hamiltonian as a function of the generalized coordinates and generalized momenta: $H = H(q_i, p_i)$, 14.3.10 aH/aH so that $dH = \sum_i p_i dq_i + \sum_i \dots dp_i$, 14.3.11 i q_i i p_i from which we see that $aH/aH = p_i$ and $\partial H/\partial p_i = q_i$ 14.3.12,13 q p In summary, then, equations 14.3.4,5,12 and 13: $aH/aH = q_i$ p which I personally find impossible to commit accurately to memory (although note that there is one dot in each equation) except when using them frequently, may be regarded as Hamilton's equations of motion. I'll refer to these equations as A, B, C and D. Note that, in the second equation, if the lagrangian is independent of the coordinate q_i , the coordinate q_i is referred to as an "ignorable coordinate". I suppose it is called "ignorable" because you can ignore it when calculating the lagrangian, but in fact a foenannt? "ignorable" coordinate is usually a very interesting coordinate indeed, because it means (look at the second equation) that the corresponding generalized momentum is conserved. Now the kinetic energy of a system is given by $T = \frac{1}{2} \sum_i m_i \dot{q}_i^2$ (for example, $m v^2$), i and the hamiltonian (equation 14.3.6) is defined as $H = \sum_i p_i \dot{q}_i - L$. For a conservative system, $L = T - V$, and hence, for a conservative system, $H = T + V$. If you are asked in an examination to explain what is meant by the hamiltonian, by all means say it is $T + V$. That's fine for a conservative system, and you'll probably get

half marks. That's 50% - a D grade, and you've passed. If you want an A+, however, I recommend equation 14.3.6. 14.4 Examples I'll do two examples by hamiltonian methods - the simple harmonic oscillator and the soap slithering in a conical basin. Both are conservative systems, and we can write

the hamiltonian as $T + V$, but we need to remember that we are regarding the hamiltonian as a function of the generalized coordinates and momenta. Thus we shall generally write translational kinetic energy as $p^2/(2m)$ rather than as $\frac{1}{2}mv^2$, and rotational kinetic energy as $L^2/(2I)$ rather than as $\frac{1}{2}I\omega^2$. Simple harmonic oscillator 12 The potential energy is $\frac{1}{2}kx^2$, so the hamiltonian is $H = \frac{p^2}{2m} + \frac{1}{2}kx^2$. 2m From equation D, we find that $\dot{x} = p/m$, from which, by differentiation with respect to the time, $\ddot{x} = \dot{p}/m$. And from equation C, we find that $\ddot{x} = -kx$. Hence we obtain the equation of motion $m\ddot{x} = -kx$. Conical basin We refer to section 13.6 of chapter 13. 12 $r^2 \dot{\theta} = h/m$, $V = mgr \cos \theta$, $L = m(r^2 \dot{\theta} + r \sin \theta \dot{\phi})$, $H = \frac{1}{2}m(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) + mgr \cos \theta$. But, in the hamiltonian formulation, we have to write the hamiltonian in terms of the generalized momenta, and we need to know what they are. We can get them from the lagrangian and equation A applied to each coordinate in turn. Thus $\frac{\partial L}{\partial \dot{r}} = m\dot{r}$ and $p_\theta = mr^2 \dot{\theta}$. 14.4.1, 2 Pr ah a Thus the hamiltonian is

$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + mgr \cos \theta$. 14.4.3 $2m$ $2mr^2 \sin^2 \theta$ Now we can obtain the equations of motion by applying equation D in turn to r and θ and then equation C in turn to r and θ : $\dot{r} = p_r/m$, $\dot{\theta} = p_\theta/(mr^2)$, 14.4.4 $p_r = m\dot{r}$, 14.4.5 $63p_\theta = mr^2 \dot{\theta}$, $H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + mgr \cos \theta$, 14.4.6 $t^3 - 63r^3 \sin^2 \theta$ $H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + mgr \cos \theta$. Equations 14.4.2 and 7 tell us that $mr^2 \sin^2 \theta \dot{\phi}$ (b is constant and therefore that $r^2 \dot{\phi}$ is constant, $= h$, say. 14.4.8 This is one of the equations that we arrived at from the lagrangian formulation, and it expresses constancy of angular momentum. By differentiation of equation 14.4.1 with respect to time, we see that the left hand side of equation 14.4.6 is $m\ddot{r}$. On the right hand side of equation 14.4.6, we have p_r , which is constant and equal to $mh \sin^2 \theta$ or. Equation 14.4.6 therefore becomes $h^2 \sin^2 \theta \ddot{\theta} - \frac{1}{3}g \cos \theta$, which we also derived from the lagrangian formulation. 14.4.9

CHAPTER 15 SPECIAL RELATIVITY 15.1. Introduction Why a chapter on relativity in a book on "classical mechanics"? A first excuse might be that the phrase "classical mechanics" is used by different authors to mean different things. To some, it means "pre-relativity"; to others it means "pre-quantum mechanics". For the purposes of this chapter, then, I mean the latter, so that special relativity may fairly be included in "classical" mechanics. A second excuse is that, apart from one brief foray into an electromagnetic problem, this chapter deals only with mechanical, kinematic and dynamical problems,

and therefore deals with only a rather restricted part of relativity that can be dealt with conveniently in a single chapter of classical mechanics rather than in a separate book. This is in fact a quite substantial restriction, because electromagnetic theory plays a major role in special relativity. It was in fact difficulties with electromagnetic theory that led Einstein to the special theory of relativity. Indeed, Einstein's theory of relativity was introduced to the world in a paper with the title *Zur Elektrodynamik bewegter Körper* (On the Electrodynamics of Moving Bodies), *Annalen der Physik*, 17, 891 (1905). The phrase "special" relativity deals with the transformations between reference frames that are moving with respect to each other at constant relative velocities. Reference frames that are accelerating or rotating or moving in any manner other than at constant speed in a straight line are included as part of general relativity and are not considered in this chapter.

15.2. The Speed of Light

The speed of light is, by definition, exactly $2.997\,924\,58 \times 10^8 \text{ m/s}$, and is the same relative to all observers. This seemingly simple sentence invites several comments. First: Note that I have used the word "speed". Some writers use the word "velocity" as if it were merely a more impressive and scientific-sounding synonym for "speed". I trust that all readers of these notes know the difference and will use the word "speed" when they mean "speed", and the word "velocity" when they mean "velocity" - surely not an unreasonable demand. To say that the "velocity" of light is the same for all observers means that the direction of travel of light is the same relative to all observers. This is doubtless not at all what a writer who uses the word "velocity" intends to convey - but it is the literal (and of course quite erroneous) meaning of the assertion. Second: How can we possibly define the speed of light to have a certain exact value? Surely the speed of light is what we find it to be, and we are not free to define its value. But in fact we are allowed to do this, and the explanation, briefly, is as follows.

Over the course of history, the metre has been defined in several different ways. At one time it was a specified fraction of the circumference of Earth. Later, it was the distance between two scratches on a bar of platinum-iridium alloy held in Paris. Later still it was a specified number of wavelengths of a particular line in the spectrum of mercury, or cadmium, or argon or krypton. In our present state of technology it is far easier to measure and reproduce precise standards of frequency than it is to measure and reproduce standards of length. Because of that, the current SI (Système International) unit of time

is the SI second, which is based on the frequency of a particular transition in the spectrum of caesium, and from there, the metre is defined as the distance travelled by light in vacuo in a defined fraction of an SI second, the speed of light being assigned the exact value quoted above. Detailed discussion of the exact definitions of the units of time, distance and speed is part of the subject of metrology. That is an important and interesting subject, but it is only marginally relevant to the topic of relativity, and consequently, having quoted the exact value of the speed of light, we leave further discussion of metrology here. Third: How can the speed of light be the same relative to all observers? This assertion is absolutely central to the theory of special relativity, and it may be regarded as its fundamental and most important principle. We shall discuss it further in the remainder of the chapter.

15.3. Preparation The ratio of the speed v of a body (or a particle, or a reference frame) is often given the symbol β : $\beta = v / c$. 15.3.1 For reasons that will become apparent (I hope!) later, the range of β is usually restricted to between 0 and 1. In our study of special relativity, we shall find that we have to make frequent use of a number of functions of β . The most common of these are $\gamma = (1 - \beta^2)^{-1/2}$, 15.3.2 $k = \gamma \beta$, 15.3.3 $z = k^{-1}$, 15.3.4 $K = \gamma - 1$, 15.3.5 $\theta = -\ln[(1 + \beta)(1 - \beta)] = \tanh^{-1} \beta$. 15.3.6

$\theta = \cos^{-1} \gamma = \sin^{-1}(i\beta\gamma)$. 15.3.7 In figures XV. 1-3 I draw γ , k and θ as functions of β . The functions γ and k go from 1 to c as β goes from 0 to 1; z , K and θ go from 0 to c . The function θ is imaginary. Many - one might even say most - problems in special relativity (including examination and homework questions!) amount, when stripped of their verbiage, to the following: "Given one of the quantities β , γ , k , z , K , θ , calculate one of the others." Thus I would suggest that, even before you have any idea what these quantities mean, you might write a program for your computer (or programmable calculator) such that, when you enter any one of the real quantities, the computer will instantly return all six of them. This will save you, on future occasions, from having to remember the exact formulas or having to bother with tedious arithmetic, so that you can concentrate your mind on understanding the relativity.

IO 9 8 7 6 5 4 3 2 1 o o

FIGURE XV. 1

I I I I I I I I 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9

4 20 18 16 14 12 10 8 6 4 2 0 0 2.5 2 1.5 0.5 0 0 k 0.1 0.2

FIGURE XV.2 I I

0.4 0.5 I I I I 0.6 0.7 0.8 0.9

FIGURE XV.3 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

Just for future reference, I tabulate here the relations between these various quantities. This has involved some algebra and typesetting; I don't think there are any mistakes, but I hope some reader might check through them all carefully and will let me know (universe.uvm.uvic.ca) if he or she finds any.

$$I = \frac{1}{k} \frac{2}{1} \frac{z}{(z+2)} \frac{1}{(1+2)} e^{20} \frac{1}{1} = \tanh(b) \text{ or } -i \tan \theta$$

$$\frac{1}{3} = \frac{1}{2} \frac{k^2+1}{(z+1)^2+1} \frac{K+I}{e^{20+1}} \frac{1}{1} \frac{k^2+1}{(z+1)^2+1} \frac{K+I}{\cosh d} \text{ or } -(e^d + e^{-d}) : \cos f$$

$$k : , : \frac{1}{4} \frac{21}{z+1} : K+I + K(1+2) : e^o : e^{iO} = : / + \frac{1}{3} : y + x f^2 : k : x + x + 2) : ?$$

$$: e^{i0} \frac{1}{(k+1)^2} \frac{(z+2)^2}{(e^{\frac{1}{2}} + 1)^2} \frac{K}{1} \frac{1}{32} \frac{1}{1} = y \frac{1}{1} = \cos \theta \frac{1}{2} \frac{k^2}{(z+1)^2}$$

$$2e^{\frac{1}{2}} d = \tanh \frac{1}{3} \text{ or } \ln(l + \frac{1}{3}) = \cosh 1y \text{ or } \ln(y + y^{21/7}) = \ln k = \ln(z+1) =$$

$$\ln(K+1 + K(K^{1/2})) = \ln f = \frac{1}{\ln(l + \frac{1}{3})} = \ln(y + \frac{1}{2}) = \ln k = \ln(z+1) =$$

$$\ln[K+1 + K(K^{1/2})] = \ln d ?$$

15.4. Speed is Relative. The Fundamental Postulate of Special Relativity. You are sitting in a railway carriage (or a railroad car, if you prefer the term). The windows and curtains are closed and you cannot see outside. You are asked to measure the constant speed of the carriage along its tracks. You try a number of experiments. You measure the period of a simple pendulum. You slide a puck and roll a ball down an inclined plane. You throw a ball vertically up in the air and catch it as it comes down. You throw it up at an angle and you watch it describe a graceful parabola. You cause billiard balls to collide on the billiards table thoughtfully provided in your carriage. You experiment with a torsion pendulum. You stand a pencil on its end and you watch it as it falls to a horizontal position.

All your careful work is to no avail. None of them tells you what speed you are moving at, or even if you are moving at all. After exhausting all mechanical experiments you can think of, you are led to the conclusion: It is impossible to determine the speed of motion of a uniformly-moving reference frame by means of any mechanical experiment performed within that frame. Frustrated, you open a curtain on one side of the carriage. You look out and you see that there is another train on the line next to you. It appears to be moving backwards. Or are you moving forwards? Or are you both moving in the same direction but at different speeds? You still can't tell. You move to the other side of the carriage and open the curtain there. This time you see the station platform, and the station platform is moving backwards. Or are you moving forwards? (Those of you who have not done much travel by train may not appreciate just how very strong the impression can be that the platform is moving.) What does it mean, anyway, to say that it is you that is moving rather than the platform? The following story is not true, but it ought

to be. (It is an "apocryphal" story.) Einstein was travelling by train across Canada. Halfway across the Prairies he leant across and tapped on the knee of his fellow passenger and asked: "Excuse me, mein Herr, I might, but does Regina stop at this train?" You are about to conclude that it is not possible by any means, whether by experiment or by observation, to determine the speed of your reference frame, or even whether it is moving or stationary. But not so hasty! I am about to invent a speedometer, which I intend to patent and to use to make myself rich. I am going to use my invention to measure the speed of our train - without even looking out of the window! We shall set up two long parallel glass rods in the middle of the corridor, parallel to the railway lines and to the velocity of the train. We shall suspend the rods horizontally, side by side from a common support, and we shall rub each of them with a silken handkerchief, so that each of them bears an electrostatic charge of $C\ m$. They will repel each other with an electrostatic force per unit length of $F_e = \frac{1}{4\pi\epsilon_0} \frac{C^2}{r}$, where r is their distance apart, and consequently they will hang out of the vertical - see figure XV.4.

FIGURE XV.4 Now see what happens when the train moves forward at speed v . Each rod, bearing a charge λ per unit length, is now moving forward at speed v , and therefore each rod constitutes an electric current λv . Therefore, by Ampere's law, in addition to the Coulomb repulsion, they will experience a magnetic attraction per unit length equal to $F_m = \frac{\mu_0}{4\pi} \frac{(\lambda v)^2}{r}$. The net repulsive force per unit length is now $F = \frac{1}{4\pi\epsilon_0} \frac{C^2}{r} - \frac{\mu_0}{4\pi} \frac{(\lambda v)^2}{r}$. This is a little less than it was when the train was stationary, so the angle between the suspending strings is a little less, as shown in figure XV.5. It might be noted that the force between the strings is reduced to zero (and the angle also becomes zero) when the train is travelling at a speed $1/\sqrt{\epsilon_0\mu_0}$ %. We remember from electromagnetic theory that the permeability of free space is $\mu_0 = 4\pi \times 10^{-7}\ H\ m$ and that the permittivity ϵ_0 is

$8.8542 \times 10^{-12}\ F\ m$; consequently the force and the angle drop to zero and the strings hang vertically, when the train is moving at a speed of $2.998 \times 10^8\ m\ s^{-1}$. FIGURE XV.5 To complete my invention, I am now going to attach a protractor to the instrument, but instead of marking the protractor in degrees, I am going to calibrate it in miles per hour, and my speedometer is now ready for use (figure XV.6). FIGURE XV.6

You now have a choice. Either: i. You can choose to believe that the speedometer will work and you can accompany me to the patent office to see if they will grant a patent for this invention, which will measure the speed of a train without reference to any external reference frame. If you choose to believe this, there is no need for you to read the remainder of the chapter on special relativity. Or ii. You can say that it defies common sense to believe that it is possible to determine whether a given reference frame is moving or stationary, let alone to determine its speed. Common sense dictates that it is impossible to determine the speed of motion of a uniformly-moving reference frame by any means whatever, whether by a mechanical or electrical or indeed any experiment performed entirely or partially within that frame, or even by reference to another frame. Your common sense, then, leads you - as it should - to the fundamental principle of special relativity. Whereas some people protest that relativity "defies common sense", in fact relativity is common sense, and its predictions (such as your prediction that my speedometer will not work) are exactly what common sense would lead you to expect.

15.5. The Lorentz Transformations For the remainder of this chapter I am taking, as a fundamental postulate, that it is impossible to determine the speed of motion of a uniformly-moving reference frame by any means whatever, whether by a mechanical or electrical or indeed any experiment performed entirely or partially within that frame, or even by reference to another frame and consequently am I choosing to believe that my speedometer will not work. If it is impossible by any electrical experiment to determine our speed, we must assume that all the electromagnetic equations that we know, not just the ones that we have quoted, but indeed Maxwell's equations, which embrace all electromagnetic phenomena, are the same in all uniformly-moving reference frames. One of the many predictions of Maxwell's equations is that electromagnetic radiation (which includes light) travels at a speed

$c = 1/x/0$ %. 15.5.1 Presumably neither the permeability nor the permittivity of space changes merely because we believe that we are travelling through space - indeed it would defy common sense to suppose that they would. Consequently, our acceptance of the fundamental principle of special relativity is equivalent to accepting as a fundamental postulate that the speed of light in vacuo is the same for all observers in uniform relative motion. We shall take anything other than this to be an outrage against common sense -

though acceptance of the principle will require a careful examination of our ideas concerning the relations between time and space. Let us imagine two reference frames, and ' is moving to the right (positive x- direction) at speed v ; relative to . (For brevity, I shall from time to time refer to as the "stationary" frame, in the hope that this liberty will not lead to misunderstanding.) At time $t = t' = 0$ the two frames coincide, and at that instant someone strikes a match at the common origin of the two frames. At a later time, which I shall call t if referred to the frame , and t' if referred to ' , the light from the match forms a spherical wavefront travelling radially outward at speed c from the origin O of , and the equation to this wavefront, when referred to the frame , is $x^2 + y^2 + z^2 - c^2 t^2 = 0$. 15.5.2 Referred to Z' , it also travels outward at speed c from the origin O' of Z' , and the equation to this wavefront, when referred to the frame ' , is $X'^2 + y'^2 + z'^2 - C^2 t'^2 = 0$. 15.5.3 Most readers will accept, I think, that $y = y'$ and $z = z'$. Some formal algebra may be needed for a rigorous proof, but that would distract from our main purpose of finding a transformation between the primed and unprimed coordinates such that $X'^2 - C^2 t'^2 = X^2 - C^2 t^2$ 15.5.4 It is easy to show that the "Galilean" transformation $x' = x - vt$, $t' = t$ does not satisfy this equality, so we shall have to try harder. Let us seek linear transformations of the form $x' = Ax + Bt$, 15.5.5 $t' = Cx + Dt$, 15.5.6 which satisfy equation 15.5.4.

11 $x' = Ax + Bt$ We have $-t' = Cx + Dt$ 15.5.7 and, by inversion, $x = Dx' - Ct' t$ $Ax' - Ct'$ 15.5.8 Consider the motion of O' relative to Z and to Z' . We have $x/t = v$ and $x' = 0$. $\therefore v = -B/A$. 15.5.9 Consider the motion of O relative to Z' and to Z . We have $x'/t' = -v$ and $x = 0$. $\therefore -v = B/D$. 15.5.10 From these we find that $D = A$ and $B = -Av$, so we arrive at $x' = A(x - vt)$ and $t' = Cx + At$. 15.5.11 15.5.12 On substitution of equations 15.5.11 and 15.5.12 into equation 15.5.4, we obtain $A^2(x-vt)^2 - c^2(Cx-At)^2 = x^2 - c^2 t^2$. Equate powers of t^2 to obtain $1 - A^2 v^2 / c^2 = 1 - C^2$. Equate powers of xt to obtain 15.5.13 15.5.14 $C = -Av/A$ 15.5.15 c Equating powers of x^2 produces no new information. We have now determined A , B , C and D , and we can substitute them into equations 15.5.5 and 15.5.6, and hence we arrive at $x' = A(x - vt)$ 15.5.16

12 and $t' = A(t - vx/c^2)$. 15.5.17 These, together with $y = y'$ and $z = z'$, constitute the Lorentz transformations, which, by suitable choice of axes, guarantee the invariance of the speed of light in all reference frames moving

at constant velocities relative to one another. To express x and t in terms of x' and t' , you may, if you are good at algebra, solve equations 15.5.16 and 15.5.17 simultaneously for x' and t' , or, if instead, you have good physical insight, you will merely reverse the sign of v and interchange the primed and unprimed quantities. Either way, you should obtain $x = \gamma(x' + vt')$ 15.5.18 and $t = \gamma(t' + vx'/c^2)$. 15.5.19 15.6. But This Defies Common Sense At this stage one may hear the protest: "But this defies common sense!". One may hear it again as we encounter several predictions of the invariance of the speed of light and of the Lorentz transformations. But, if you have read this far, it is too late to make such protest. You have already, at the end of Section 15.4, made your choice, and you then decided that it defies common sense to suppose that one can somehow determine the speed of a reference frame by some experiment or observation. You rejected that notion, and it was the application of common sense, not its abandonment, that led us into the Lorentz transformations and the invariance of the speed of light. There may be other occasions when we are tempted to protest "But this defies common sense!", and it is therefore always salutary to recall this. For example, we shall later learn that if a train is moving at speed V relative to the station platform, and a passenger is walking towards the front of the train at a speed v relative to the train, then, relative to the platform, he is moving at a speed just a little bit less than $V + v$. When we protest, we are often presented with an "explanation" along the following lines: In every day life, trains do not move at speeds comparable to the speed of light, nor do walking passengers. Therefore, we do not notice that the combined speed is a little bit less than $V + v$. After all, if $V = 60$ mph and $v = 4$ mph, the combined speed is 0.999 999 999 5 x 64 mph. The formula $V + v$ is just an approximation, we are told, and we have the erroneous impression that the combined speed is exactly $V + v$ only because we are accustomed, in daily life, to experiencing speeds that are small compared with the speed of light. This explanation somehow does not seem to be satisfactory - and nor should it, for it is not a correct explanation. It seems to be an explanation invented for the benefit of the nonscientific layman - but nothing is ever made easy to understand by giving an incorrect

13 explanation under the pretence of "simplifying" something. It is not correct merely to say that the Galilean transformations are just an "approximation" to the "real" transformations. The problem is that it is

exceedingly difficult - perhaps impossible - to describe exactly what is meant by "distance" and "time interval". It is almost as difficult as describing colours to a blind person, or even describing your sensation of the colour red to another seeing person. We have no guarantee that every person's perception of colour is the same. The best that can be done to describe what we mean by distance and time interval is to define how distances and times transform between reference frames. The Lorentz transformations, which we have adopted in order to make it meaningless to discuss the absolute velocity of a reference frame, amount to a useful working definition of the meanings of space and time. Once we have adopted this definition, "common sense" no longer comes into the matter. There is no longer a mystery which our minds cannot quite grasp; from this point on it merely becomes a matter of algebra as to how a measurement of length or of time interval, or of speed, or of mass, as appropriately defined, transforms when referred to one reference or to another. There is no impossible feat of imagination to be done.

15.7 The Lorentz Transformation as a Rotation

The Lorentz transformation can be written $x'_1 = \gamma(x_1 - v x_4)$, $x'_2 = x_2$, $x'_3 = x_3$, $x'_4 = \gamma(x_4 - \frac{v}{c^2} x_1)$ where $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = ct$, and similarly for primed quantities. Please don't just take my word for this; multiply the matrices, and verify that this equation does indeed represent the Lorentz transformation. You could, if you wish, also write this for short: $x' = \Lambda x$.

15.7.2 Another way of writing the Lorentz transformation is $X' = \Lambda X$ where $X = (x_1, x_2, x_3, x_4)$ and $X' = (x'_1, x'_2, x'_3, x'_4)$.

15.7.3 where $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = ct$, and similarly for primed quantities.

14 Some people prefer one version; others prefer the other. In any case, a set of four quantities that transforms like this is called a 4-vector. Those who dislike version 15.7.1 dislike it because of the introduction of imaginary quantities. Those who like version 15.7.1 point out that the expression $[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c \Delta t)^2]^{1/2}$ (the "interval" between two events) is invariant in four-space - that is, it has the same value in all uniformly-moving reference frames, just as the distance between two points in three-space, $[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]^{1/2}$, is independent of the position or orientation of any reference frame. In version 15.7.3, the invariant interval is $[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (\Delta x_4)^2]^{1/2}$. Those who prefer version 15.7.1 dislike the minus sign in the expression for the interval. Those who prefer version 15.7.3 dislike the imaginary quantities of version 15.7.1. For the time being, I am going to omit y and z , so that I can concentrate my

attention on the relations between x and t . Thus I am going to write 15.7.1 as $x = \gamma(x' + vt't)$ and equation 15.7.3 as $ct = \gamma(ct' + \beta x')$. Readers may notice how closely equation 15.7.4 resembles the equation for the transformation of coordinates between two reference frames that are inclined to each other at an angle. (See Celestial Mechanics Section 3.6.) Indeed, if we let $\cos \theta = \gamma$ and $\sin \theta = i\beta$, equation 15.7.4 becomes $ict' = (-\sin \theta \cos \theta ict + \cos^2 \theta x)$. The matrices in equations 15.7.1, 15.7.4 and 15.7.6 are orthogonal matrices and they satisfy each of the criteria for orthogonality described, for example, in Celestial Mechanics Section 3.7. We can obtain the converse relations (i.e. we can express x and t in terms of x' and t') by interchanging the primed and unprimed quantities and either reversing the sign of β or of θ or by interchanging the rows and columns of the matrix. There is a difficulty in making the analogy between the Lorentz transformation as expressed by equation 15.7.4 and rotation of axes as expressed by equation 15.7.6 in that, since $\gamma > 1$, θ is an imaginary angle. (At this point you may want to reach for your ancient, brittle, yellowed notes on complex numbers and hyperbolic functions.) Thus $\theta = i\phi$, and for $\gamma > 1$, this means that $\theta = i \cosh^{-1} \gamma = i \ln(\gamma + \beta\gamma)$. And

$\sin(i\phi) = i \sinh \phi = i \ln(\gamma + \beta\gamma)$. Either of these expressions reduces to $\theta = i \ln[\gamma(1 + \beta)]$. Perhaps a yet more convenient way of expressing this is 15.7.7 For example, if $\beta = 0.8$, $\theta = 1.0986i$, which might be written (not necessarily particularly usefully) as $i \times 62^\circ 57'$. At this stage, you are probably thinking that you much prefer the version of equation 15.7.5, in which all quantities are real, and the expression for the interval between two events is $[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (\Delta ct)^2]^{1/2}$. The minus sign in the expression is a small price to pay for the realness of all quantities. Equation 15.7.5 can be written $ct' = \cosh \phi \cosh \phi ct - \sinh \phi \cosh \phi x$ 15.7.8 where $\cosh \phi = \gamma$, $\sinh \phi = \beta\gamma$, $\tanh \phi = \beta$. On the face of it, this looks much simpler. No messing around with imaginary angles. Yet this formulation is not without its own set of difficulties. For example, neither the matrix of equation 15.7.5 nor the matrix of equation 15.7.8 is orthogonal. You cannot invert the equation to find x and t in terms of x' and t' merely by interchanging the primed and unprimed symbols and interchanging the rows and columns. The converse of equation 15.7.8 is in fact $ct = \sinh \phi \cosh \phi ct' + \cosh \phi \cosh \phi x'$ 15.7.9 which can also (understandably!) be written $ct = \sinh(-\phi) \cosh(-\phi) ct' + \cosh(-\phi) \cosh(-\phi) x'$ 15.7.10 which demands as much skill in handling hyperbolic functions as

the other formulation did in handling complex numbers. A further problem is that the formulation 15.7.5 does not allow the analogy between the Lorentz transformation and the rotation of axes. You take your choice. It may be noticed that the determinants of the matrices of equations 15.7.5 and 15.7.8 are each unity, and it may therefore be thought that each matrix is orthogonal and that its reciprocal is its transpose. But this is not the case, for the condition that the determinant is zero is not a sufficient condition for a matrix to be orthogonal. The necessary tests are

16 summarized in Celestial Mechanics, Section 3.7, and it will be found that several of the conditions are not satisfied. 15.8 Timelike and Spacelike 4-
 17ectors I am going to refer some events to a coordinate system whose origin is here and now and which is moving at the same velocity as you happen to be moving. In other words, you are sitting at the origin of the coordinate system, and you are stationary with respect to it. Let us suppose that an event A occurs at the following coordinates referred to this reference frame, in which the distances x , y , z are expressed in light-years (lyr) the time t is expressed in years (yr). $x = 2$ $y = 3$ $z = 7$ $t = -1$ A "light-year" is a unit of distance used when describing astronomical distances to the layperson, and it is also useful in describing some aspects of relativity theory. It is the distance travelled by light in a year, and is approximately 9.46×10^{15} m or 0.307 parsec (pc). Event A, then, occurred a year ago at a distance of $x_f = 7.87$ lyr, when referred to this reference frame. Note that, if referred to a reference frame that coincides with this one at $t = 0$, but is moving with respect to it, all four coordinates might be different, and the distance $\sqrt{x^2 + y^2 + z^2}$ and the time of occurrence would be different, but, according to the way in which we have defined space and time by the Lorentz transformation, the quantity $\sqrt{x^2 + y^2 + z^2 - c^2 t^2}$ would be the same. Imagine now a second event, B, which occurs at the following coordinates: $x_2 = 5$ $y_2 = 8$ $z_2 = 10$ $t_2 = +2$ That is to say, when referred to the same reference frame, it will occur in two years' time at a distance of $\sqrt{x^2 + y^2 + z^2} = 13.75$ lyr. The 4-vector $s = B - A$ connects these two events, and the magnitude s of s is the interval between the two events. Note that the distance between the two events, when referred to our reference frame, is $\sqrt{(5-2)^2 + (8-3)^2 + (10-7)^2} = 6.56$ lyr. The interval between the two events is $\sqrt{(5-2)^2 + (8-3)^2 + (10-7)^2 - (2+1)^2} = 5.83$ lyr, and this is independent of the velocity of the reference frame. That is, if we "rotate" the reference frame, it obviously makes no difference to the interval between the two events, which

is invariant. As another example, consider two events A and B whose coordinates are $x = 2$ $y = 5$ $z = 3$ $t = -2$

$x_2 = 3$ $y_2 = 7$ $z_2 = 4$ $t_2 = +6$ with distances, as before, expressed in lyr, and times in yr. Calculate the interval between these two events - i.e. the magnitude of the 4-vector connecting them. If you carry out this calculation, you will find that $s^2 = -58$, so that the interval s is imaginary and equal to $7.62i$. So we see that some pairs of events are connected by a 4-vector whose magnitude is real, and other pairs are connected by a 4-vector whose magnitude is imaginary. There are differences in character between real and imaginary intervals, but, in order to strip away distractions, I am going to consider events for which $y = z = 0$. We can now concentrate on the essentials without being distracted by unimportant details. Let us therefore consider two events A and B whose coordinates are $x = 2$ $t = -2$ $x_2 = 3$ $t_2 = +6$ yr. These events and the 4-vector connecting them are shown in figure XV.7. Event A happened two years ago (referred to our reference frame); event B will occur (also referred to our reference frame) in six years' time. The square of the interval between the two events (which is invariant) is -63 lyr^2 , and the interval is imaginary. If someone wanted to experience both events, he would have to travel only 1 lyr (referred to our reference frame), and he could take his time, for he would have eight years (referred to our reference frame) in which to make the journey to get to event B in time. He couldn't totally dawdle, however; he would have to travel at a speed of at least $\frac{1}{8}$ times the speed of light, but that's not extremely fast for anyone well versed in relativity. FIGURE XV.7 x B Let's look at it another way. Let's suppose that event A is the cause of event B. This means that some agent must be capable of conveying some information from A to B at a speed at least equal to $\frac{1}{8}$ times the speed of light. That may present some technical problems, but it presents no problems to our imagination.

18 You'll notice that, in this case, the interval between the two events - i.e. the magnitude of the 4-vector connecting them - is imaginary. A 4-vector whose magnitude is imaginary is called a timelike 4-vector. There is quite a long time between events A and B, but not much distance. Now consider two events A and B whose coordinates are $x = 2$ $t = -1$ $x_2 = 7$ $t_2 = +3$ yr. The square of the magnitude of the interval between these two events is $+9 \text{ lyr}^2$, and the interval is real. A 4-vector whose magnitude is real is called a

spacelike 4-vector. It is shown in figure XV.8. x B A FIGURE XV.8 Perhaps I could now ask how fast you would have to travel if you wanted to experience both events. They are quite a long way apart, and you haven't much time to get from one to the other. Or, if event A is the cause of event B, how fast would an information-carrying agent have to move to convey the necessary information from A in order to instigate event B? Maybe you have already worked it out, but I'm not going to ask the question, because in a later section we'll find that two events A and B cannot be mutually causally connected if the interval between them is real. Note that I have said "mutually"; this means that A cannot cause B, and B cannot cause A. A and B must be quite independent events; there simply is too much space in the interval between them for one to be the cause of the other. It does not mean that the two events cannot have a common cause. Thus, figure XV.9 shows two events A and B with a spacelike interval between them (very steep) and a third event C such the intervals CA and CB (very shallow) are timelike. C could easily be the cause of both A and B; that is, A and B could have a common cause. But there can be no mutual causal connection between A and B. (It might be noted parenthetically that Charles Dickens temporarily nodded when he chose

19 the title of his novel *Our Mutual Friend*. He really meant our common friend. C was a friend common to A and to B. A and B were friends mutually to each other.) Distance B C FIGURE XV.9 Time Exercise. The distance of the Sun from Earth is 1.496×10^{10} m. The speed of light is 2.998×10^8 m/s. How long does it take for a photon to reach Earth from the Sun? Event A: A photon leaves the Sun on its way to Earth. Event B: The photon arrives at Earth. What is the interval (i.e. s in 4-space) between these two events? 15.9 The FitzGerald-Lorentz Contraction This is sometimes described in words something like the following: If a measuring-rod is moving with respect to a "stationary" observer, it "appears" to be shorter than it "really" is. This is not a very precise statement, and the words that I have placed in inverted commas call for some clarification. We have seen that, while the interval between two events is invariant between reference frames, the distance between two points (and hence the length of a rod) depends on the coordinate frame to which the points are referred. Let us now define what we mean by the length of a rod. Figure XV. 10 shows a reference frame, and a rod lying parallel to the x-axis. For the moment I am not specifying whether the rod is moving with respect

to the reference frame, or whether it is stationary.

20 Y FIGURE XV. 10 x_1 x_2 x Let us suppose that the x -coordinate of the left-hand end of the rod is x , and that, at the same time referred to this reference frame, the x -coordinate of the right-hand end is x_2 . The length l of the rod is defined as $l = x_2 - x$. That could scarcely be a simpler statement - but note the little phrase "at the same time referred to this reference frame". That simple phrase is important. Now let's look at the FitzGerald-Lorentz contraction. See figure XV. 11. Y X FIGURE XV. 11 The are two reference frames, Z and Z' . The frame Z' is moving to the right with respect to Z with speed v . A rod is at rest with respect to the frame Z' , and is therefore moving to the right with respect to Z at speed v . In my younger days I often used to travel by train, and I still like to think of railway trains whenever I discuss relativity. Modern students usually like to think of spacecraft, presumably because they are more accustomed to this mode of travel. In the very early days of railways, it was customary for the stationmaster to wear top hat and tails. Those days are long gone, but, when thinking about the FitzGerald-Lorentz contraction, I like to

21 think of as being a railway station in which there resides a stationmaster in top hat and tails, while Z' is a railway train. The length of the rod, referred to the frame Z' , is $l' = x'_2 - x'_1$, in what I hope is obvious notation, and of course these two coordinates are determined at the same time referred to Z' . The length of the rod referred to a frame in which it is at rest is called its proper length. Thus l' is the proper length of the rod. Now it should be noted that, according to the way in which we have defined distance and time by means of the Lorentz transformation, although x'_2 and x'_1 are measured simultaneously with respect to Z' , these two events (the determination of the coordinates of the two ends of the rod) are not simultaneous when referred to the frame Z (a point to which we shall return in a later section dealing with simultaneity). The length of the rod referred to the frame Z is given by $l = x_2 - x_1$, where these two coordinates are to be determined at the same time when referred to Z . Now equation 15.5.16 tells us that $x_2 = \gamma(x'_2 + vt')$ and $x_1 = \gamma(x'_1 + vt')$. (Readers should note this derivation very carefully, for it is easy to go wrong. In particular, be very clear what is meant in these two equations by the symbol t . It is the single instant of time, referred to Z , when the coordinates of the two ends are determined simultaneously with respect to Z .) From these

we reach the result: $\gamma = 1/\sqrt{1 - v^2/c^2} = 3.2026$. 15.9.1 This is the FitzGerald-Lorentz contraction. It is sometimes described thus: A railway train of proper length 100 yards is moving past a railway station at 95% of the speed of light ($\gamma = 3.2026$.) To the stationmaster the train "appears" to be of length 31.22 yards; or the stationmaster "thinks" the length of the train is 31.22 yards; or, "according to" the stationmaster the length of the train is 31.22 yards. This gives a false impression, as though the stationmaster is under some sort of misapprehension concerning the length of the train, or as if he is labouring under some sort of illusion, and it introduces some sort of unnecessary "mystery" into what is nothing more than simple algebra. In fact what the stationmaster "thinks" or "asserts" is entirely irrelevant. Two correct statements are: 1. The length of the train, referred to a reference frame in which it is at rest - i.e. the proper length of the train - is 100 yards. 2. The length of the train when referred to a frame with respect to which it is moving at a speed of $0.95c$ is 31.22 yards. And that is all there is to it. Any phrase such as "this observer thinks that" or "according to this observer" should always be interpreted in this manner. It is not a matter of what an observer "thinks". It is a matter of which frame a measurement is referred to. Nothing more, nothing less.

22 ict ict' FIGURE XV. 12 It is possible to describe the Lorentz-FitzGerald contraction by interpreting the Lorentz transformations as a rotation in 4-space. Whether it is helpful to do so only you can decide. Thus figure XV. 12 shows Z and Z' related by a rotation in the manner described in section 15.7. The thick continuous line shows a rod oriented so that its two ends are drawn at the same time with respect to Z'. Its length is, referred to Z', l' , and this is its proper length. The thick dotted line shows the two ends at the same time with respect to Z. Its length referred to Z is $I = l'/\cos\theta$. And, since $\cos\theta = \sqrt{1 - v^2/c^2}$, which is greater than 1, this means that, in spite of appearances in the figure, $I < l'$. The figure is deceptive because, as discussed in section 15.7, θ is imaginary. As I say, only you can decide whether this way of looking at the contraction is helpful or merely confusing. It is, however, at least worth looking at, because I shall be using this concept of rotation in a forthcoming section on simultaneity and order of events. Illustrating the Lorentz transformations as a rotation like this is called a Minkowski diagram. 15.10 Time Dilation We imagine the same railway train Z' and the same railway station Z as in the previous section except that, rather than measuring a length

referred to the two reference frames, we measure the time interval between two events. We'll suppose that a passenger in the railway train Z' claps his hands twice. These are two events which, when referred to the reference frame Z , take place at the same place when referred to this reference frame. Let the instants of time when the two events occur, referred to Z , be t_1 and t_2 . The time interval T is defined as $t_2 - t_1$. But the Lorentz transformation is $t = \gamma(t' + vx'/c^2)$, and so the time interval when referred to Z is $T = \gamma T'$.

15.10.1

23 This is the dilation of time. The situation is illustrated by a Minkowski diagram in figure XV.13. While it is clear from the figure that $T = T' \cosh \theta$ and therefore that $T = \gamma T'$, it is not so clear from the figure that this means that T is greater than T' - because $\cosh \theta > 1$ and θ is imaginary. ict' ict X X FIGURE XV. 13 Thus, let us suppose that a passenger on the train holds a 1-metre measuring rod (its length in the direction of motion of the train) and he claps his hands at an interval of one second apart. Let's suppose that the train is moving at 98% of the speed of light ($\gamma = 5.025$). In that case the stationmaster thinks that the length of the rod is only 19.9 cm and that the time interval between the claps is 5.025 seconds. I deliberately did not word that last sentence very well. It is not a matter of what the stationmaster or anyone else "thinks" or "asserts". It is not a matter that the stationmaster is somehow deceived into erroneously believing that the rod is 19.9 cm long and the claps 5.025 seconds apart, whereas they are "really" 1 metre long and 1 second apart. It is a matter of how length and time are defined (by subtracting two space coordinates determined at the same time, or two time coordinates at the same place) and how space- time coordinates are defined by means of the Lorentz transformations. The length is 19.9 cm, and the time interval is 5.025 seconds when referred to the frame Z . It is true that the proper length and the proper time interval are the length and the time interval referred to a frame in which the rod and the clapper are at rest. In that sense one could loosely say that they are "really" 1 metre long and 1 second apart. But the Lorentz contraction and the time dilation are not determined by what the stationmaster or anyone else "thinks". Another way of looking at it is this. The interval s between two events is clearly independent of the orientation any reference frames, and is the same when referred to two reference frames that may be inclined to each other. But the components of the vector

24 joining two events, or their projections on to the time axis or a space axis are not at all expected to be equal. By the way, in section 15.3 I urged you to write a computer or calculator programme for the instant conversion between the several factors commonly encountered in relativity. I still urge it. As soon as I typed that the train was travelling at 98% of the speed of light, I was instantly able to generate $\diamond$. You need to be able to do that, too. 15.11 The Twins Paradox During the late 1950s and early 1960s there was great controversy over a problem known as the "Twins Paradox". The controversy was not confined to within scientific circles, but was argued, by scientists and others, in the newspapers, magazines and many serious journals. It goes something like this: There are two 20-year-old twins, Albert and Betty. Albert is a sedentary type who likes nothing better than to stay at home tending the family vineyards. His twin sister Betty is a more adventurous type, and has trained to become an astronaut. On their twentieth birthday, Betty waves a cheery au revoir to her brother and takes off on what she intends to be a brief spaceflight, at which she travels at 99.98 % of the speed of light ($\diamond = 50$). After six months by her calendar she turns back and on her 21 st birthday she arrives back home to greet her brother, only to find that he is now old and sere and has laboured, by his calendar for 50 years and is now an aged man of 71 years. If we accept what we have derived in the previous section about the dilation of time, there would seem to be no particular problem with that. It has even been argued that travel between the stars may not be an impossibility. Whereas to an Earthbound observer it may take many decades for a spacecraft to travel to a star and back, for the astronauts on board much less time has elapsed. And yet a paradox was pointed out. According to the principles of the relativity of motion, it was argued, one could refer everything to Betty's reference frame, and from that point of view one could regard Betty as being the stationary twin and Albert as the one who travelled off into the distance and returned later. Thus, it could be argued, it would be Albert who had aged only one year, while Betty would have aged fifty years. Thus we have a paradox, which is a problem which apparently gives rise to opposite conclusion depending on how it is argued. And the only way that the paradox could be resolved was to suppose that both twins were the same age when they were re-united. A second argument in favour of this interpretation that the twins were the same age when re-united points out that dilation of time arises because two events that may occur in the same place when referred to one reference frame do not occur in the same place when

referred to another. But in this case, the two events (Betty's departure and re-arrival) occur at the same place when referred to both reference frames.

25 The argument over this point raged quite furiously for some years, and a particularly plausible tool that was used was something referred to as the "k-calculus" - an argument that is, however, fatally flawed because the "rules" of the k-calculus inherently incorporate the desired conclusion. Two of the principal leaders of the very public scientific debate were Professors Fred Hoyle and Herbert Dingle, and this inspired the following letter to a weekly magazine, *The Listener*, in 1960: Sir: The ears of a Hoyle may tingle; The blood of a Hoyle may boil When Hoyle pours hot oil upon Dingle, And Dingle cold water on Hoyle. But the dust of the wrangle will settle. Old stars will look down on new soil. The pot will lie down with the kettle, And Dingle will mingle with Hoyle. So what are you, the reader expected to believe? Let us say this: If you are a student who has examinations to pass, or if you are an untenured professor who has to hold on to a job, be in no doubt whatever: The original conclusion is the canonically-accepted correct conclusion, namely that Albert has aged 50 years while his astronaut sister has aged but one. This is now firmly accepted truth. Indeed it has even been claimed that it has been "proved" experimentally by a scientist who took a clock on commercial airline flights around the world, and compared it on his return with a stay-at-home clock. For myself I have neither examinations to pass nor, alas, a job to hold on to, so I am not bound to believe one thing or the other, and I elect to hold my piece. I do say this, however - that what anyone "believes" is not an essential point. It is not a matter of what Albert or Betty or Hoyle or Dingle or your professor or your employer "believes". The real question is this: What is it that is predicted by the special theory of relativity? From this point of view it does not matter whether the theory of relativity is "true" or not, or whether it represents a correct description of the real physical world. Starting from the basic precepts of relativity, whether "true" not, it must be only a matter of algebra (and simple algebra at that) to decide what is predicted by relativity. A difficulty with this is that it is not, strictly speaking, a problem in special relativity, for special relativity deals with transformations between reference frames that are in uniform motion relative to one another. It is pointed out that Albert and Betty are not in uniform motion relative to one another, since one or the other of them has to change the direction of motion - i.e. has to accelerate. It could still be argued that,

since motion is relative, one can regard either Albert or Betty as the one who accelerates - but the response to this is that only uniform motion is relative. Thus there is no symmetry between Albert and Betty. Betty either accelerates or experiences a gravitational field (depending on whether

26 her experience is referred to Albert's or her own reference frame). And, since there is no symmetry, there is no paradox. This argument, however, admits that the age difference between Albert and Betty on Betty's return is not an effect of special relativity, but of general relativity, and is an effect caused by the acceleration (or gravitational field) experienced by Betty. If this is so, there are some severe difficulties in describing the effect under general relativity. For example, whether the general theory allows for an instantaneous change in direction by Betty (and infinite deceleration), or whether the final result depends on how she decelerates - at what rate and for how long - must be determined by those who would tackle this problem. Further, the alleged age difference is supposed to depend upon the time during which Betty has been travelling and the length of her journey - yet the portion of her journey during which she is accelerating or decelerating can be made arbitrarily short compared with the time during which she is travelling at constant speed. If the effect were to occur solely during the time when she was accelerating or decelerating, then the total length and duration of the constant speed part of her journey should not affect the age difference at all. Since this chapter deals only with special relativity, and this is evidently not a problem restricted to special relativity, I leave the problem, as originally stated, here, without resolution, for readers to argue over as they will 15.12

A, B and C A, B and C were three characters in the Canadian humorist Stephen Leacock's essay on The Human Element in Mathematics. "A, B and C are employed to dig a ditch. A can dig as much in one hour as B can dig in two..." We can ask A, B and C to come to our aid in a modified version of the twins' problem, for we can arrange all three of them to be moving with constant velocities relative to each other. It goes like this (figure XV. 14):

FIGURE XV. 14 The scenario is probably obvious from the figure. There are three events: 1. B passes A 2. B meets C 3. C passes A

27 At event 1, B and A synchronize their watches so that each reads zero. At event 2, C sets his watch so that it reads the same as B's. At event 3, C and A compare watches. I shall leave the reader to cogitate over this. The only thing

I shall point out is that this problem differs from the problem described as the Twins Paradox in two ways. In the first place, unlike in the Twins Paradox, all three characters, A, B and C are moving at constant velocities with respect to each other. Also, the first and third events occur at the same place relative to A but at different places referred to B or to C. In the twin paradox problem, the two events occur at the same place relative to both frames.

15.13 Simultaneity If the time interval referred to one reference frame can be different when referred to another reference frame (and since time interval is merely one component of a four- vector, the magnitude of the component surely depends on the orientation in four space of the four axes) this raises the possibility that there might be a time interval of zero relative to one frame (i.e. two events are simultaneous) but are not simultaneous relative to another. This is indeed the case, provided that the two events do not occur in the same place as well as at the same time. Look at figure XV. 15. ict' ict

FIGURE XV. 15 X ! x I have drawn two reference frames at an (imaginary) angle θ to each other. Think of Z as the railway station and of Z' as the railway train, and that the speed of the railway train is $c \tan \theta$. (You may have to go back to section 15.3 or 15.7 to recall the relation of θ to the

28 speed.) The thick line represents the interval between two events that are simultaneous when referred to Z, but are separated in space (one occurs near the front of the train; the other occurs near the rear). (Note also in this text that I am using the phrase "time interval" to denote the time-component of the "interval". For two simultaneous events, the time interval is zero, and the interval is then merely the distance between the two events.) While the thick line has zero component along the ict' axis, its component along the ic! axis is $l' \sin \theta$. That is, $ic(t_2 - t_1) = l' \sin \theta = l' \times i[3\frac{1}{2}]$. [371' Hence: $t_2 - t_1 = 15.13.1 \frac{c}{v}$

For example, if the events took place simultaneously 100,000 km apart in the train (it is a long train) and if the train were travelling at 95% of the speed of light ($\gamma = 3.203$; it is a fast train), the two events would be separated when referred to the railway station by 1.01 seconds. The event near the rear of the train occurred first.

15.14 Order of Events, Causality and the Transmission of Information Maybe it is even possible that if one event precedes another in one reference frame, in another reference frame the other precedes the one. In other words, the order of occurrence of events may be different in two frames. This indeed can be the case, and Minkowski diagrams (figure XV. 16) can help us to see why and in what circumstances. ict ict' ict'

..... 117:.':::' 2 z Z' ..E.. Z' 2 X' X (a)
 FIGURE XV. 16 (b) In part (a), of the two events 1 and 2, 1 occurs before 2 in either Z or Z'. (from this point on I shall use a short phrase such as "in Z" rather than the more cumbersome "when referred to the reference frame Z". But in part (b), event 1 occurs before event 2 in E, but x

29 after event 2 in Z'. One can see that there is reversal of order of events if the slope of the line joining to two events is less than the angle θ . The angle θ , it may be recalled, is an imaginary angle such that $\tan \theta = i\beta = i\gamma/c$, where γ is the relative speed of the two frames. In figure XV. 17, for simplicity I am going to suppose that event 1 occurs at the origin of both frames, and that event 2 occurs at coordinates $(t; t, i\beta t)$ in Z. The condition for no reversal of events is then evidently $i\beta t > \tan \theta = i\beta = \gamma/c$ or $\gamma < c$. 15.14.1 Now suppose that events 1 and 2 are causally connected in the sense that event 1 is the cause of event 2. For this to be the case, some signal carrying information must travel from 1 to 2. However, if event 1 is the cause of event 2, event 1 must precede event 2 in all reference frames. Thus it follows that no signal carrying information that could cause an event to occur can travel faster than the speed of light. This means, in effect, that neither mass nor energy can be transmitted faster than the speed of light. That is not quite the same thing as saying that "nothing" can be transmitted faster than the speed of light. For example a Moiré pattern formed by two combs with slightly different tooth spacings can move faster than light if one of the combs is moved relative to the other; but then I suppose it has to be admitted that in that case "nothing" is actually being transmitted - and certainly nothing that can transmit information or that can cause an event. An almost identical example would be the modulation envelope of the sum of two waves of slightly different frequencies. A well-known example from wave mechanics is that of the wave representation of a moving particle. The wave group (which is the integral of a continuous distribution of wavelengths whose extent is governed by Heisenberg's principle) moves with the particle at a sub-luminal speed, but there is nothing to prevent the wavelets within the group moving through the group at any speed. These wavelets may start at the beginning of the group and rapidly move through the group and extinguish themselves at the end. No "information" is transmitted from A to B at a speed any faster than the particle itself is moving. 15.15 Derivatives We'll pause here and establish a few derivatives just for reference and in case

we need them later. We recall that the Lorentz relations are $x = \gamma(x' + vt')$
15.15.1

30 and $t = \gamma(t' + vx'/c^2)$. From these we immediately find that $x' = \gamma(x - vt)$,
c We shall need these in future sections. 15.15.2 = 7. 15.15.3a,b,c,d Caution:
It is not impossible to make a mistake with some of these derivatives if one
allows one's attention to wander. For example, one might suppose that, since
 $ax/ax' = \gamma$, then "obviously" $ax'/ax = 1/\gamma$ - and indeed this is correct if t' is
being held constant. However, we have to be sure that this is really what we
want. The difficulty is likely to arise if, when writing a partial derivative, we
neglect to specify what variables are being held constant, and no great harm
would be done by insisting that these always be specified when writing a
partial derivative. If you want the inverses rather than the reciprocals of
equations 15.15.3a,b,c,d, the rule, as ever, is: Interchange the primed and
unprimed symbols and change the sign of v or β . For example, the reciprocal
of $\partial x' / \partial t' (On: t)$ while its inverse is $(\partial On: t) / \partial t'$ For completeness, and reference,
then, I write down all the possibilities: $(\partial x, t) / \partial (ax) t: \gamma$; $(\partial x) / \partial (-\gamma v)$; $(\partial x) t / c^2$
 $(\partial x) / \partial t: \gamma'$ 15.15.3i,j,k.1 $ax = 1/\gamma$; $= -1/\gamma v$; $- ; = 1/\gamma$. 15.15.3m,n,o,p $t x t [37$
x Now let's suppose that (x, t) , where (equations 15.15.1 and 15.15.2) of x'
and t' . Then x and t are in turn functions - $+ - \gamma + ---- ax' ax' ax' at ax c at$
15.15.4 and $\partial ax \partial at at^2 - + \gamma v at^2 + \gamma$ 15.15.5 at' $\partial \partial at' at ax at$

31 The reader will doubtless notice that I have here ignored my own advice
and I have not indicated which variables are to be held constant. It would be
worth spending a moment here thinking about this. We can write equations
15.15.4 and 15.15.5 as equivalent operators: and $ax, = \gamma + 15.15.6 = \gamma v +$
 γ 15.15.7 We can also, if we wish, find the second derivatives. Thus $a^2 a a$
 $Ox' ^2 Ox' Ox' -- + cot)$ from which we find 15.15.8 $\partial^2 / \partial a^2 \partial^2 \partial^2 \partial^2 /$
 $ax, \partial^2 + -- + . C ax at c^2 at^2$ In a similar manner we obtain 15.15.9 $= + 0x' at'$
 $. ox On; at c at^2)$ 15.15.10 and $at'^2 = \gamma^2 v^2 + 2Voxa + \gamma$ 15.15.11 The
inverses of all of these relations are to be found by interchanging the primed
and unprimed coordinates and changing the signs of v and β . 15.16 Addition
of velocities A railway train trundles towards the east at speed v , and a
passenger strolls towards the front at speed v_2 . What is the speed of the
passenger relative to the railway station? We might at first be tempted to
reply: "Why, $v + v_2$, of course." In this section we shall

32 show that the answer as predicted from the Lorentz transformations is a little less than this, and we shall develop a formula for calculating it. We have already discussed (in section 15.6) our answer to the objection that this defies common sense. We pointed out there that the answer (to the perfectly reasonable objection) that "at the speeds we are accustomed to we would hardly notice the difference" is not a satisfactory response. The reason that the resultant speed is a little less than $12 + 122$ results from the way in which we have defined the Lorentz transformations between reference frames and the way in which distances and time intervals are defined with reference to reference frames in uniform relative motion. X FIGURE XV. 17 Figure XV. 17 shows two reference frames, Z and Z', the latter moving at speed 12 with respect to the former. A particle is moving with velocity u' in Z', with components U'_x and u'_y . ("in Z'" = "referred to the reference frame Z'".) What is the velocity of the particle in Z? Let us start with the x-component. $\frac{dx}{dt} = u_x$, $\frac{dx'}{dt'} = u'_x$, $\frac{dx}{dt} = \frac{dx'}{dt'} + v \frac{dt'}{dt}$ We have: $u_x = \frac{u'_x + v}{1 + \frac{v u'_x}{c^2}}$ 15.16.1 $\frac{dx}{dt} = \frac{dx'}{dt'} + v \frac{dt'}{dt}$, $\frac{dx}{dt} = \frac{dx'}{dt'} + v \frac{dt'}{dt}$ We take the derivatives from equations 15.15.3a-d, and, writing $12/c$ for β , we obtain $U_x + U u_x = : 15.16.2 1 + U_x, 12/c$ The inverse is obtained by interchanging the primed and unprimed symbols and reversing the sign of v .

33 The y-component is found in an exactly similar manner, and I leave its derivation to the reader. The result is $u_y = \frac{u'_y}{\gamma(1 + U'_x v/c^2)}$ 15.16.3 $u_y = \frac{u'_y}{\gamma(1 + U'_x v/c^2)}$ Special cases: I. If $U'_x = u'$ and $u'_y = 0$, then $u_x = u$ and $u_y = 0$ 15.16.4a,b $1 + u'v/c^2$ II. If $U'_x = 0$ and $u'_y = u'$, then $u_x = v$ and $u_y = u'/\gamma$ 15.16.5a,b Equations 15.16.4a as written is not easy to commit to memory, though it is rather easier if we write $\beta = v/c$, $\beta' = u'/c$ and $\beta = u_x/c$. Then the equation becomes 15.16.6 $\beta = \frac{\beta' + \beta}{1 + \beta \beta'}$ [32 FIGURE XV. 18 [32 FIGURE XV. 19

34 In figure XV. 18, a train Z' is trundling with speed β (times the speed of light) towards the right, and a passenger is strolling towards the front at speed β' . The speed β of the passenger relative to the station is then given by equation 15.16.6. In figure XV.19, two trains, one moving at speed β and the other moving at speed β' , are moving towards each other. (If you prefer to think of protons rather than trains, that is fine.) Again, the relative speed β of one train relative to the other is given by equation 15.16.6. Example. A train trundles to the right at 90% of the speed of light relative to Z, and a passenger strolls to the right at 15% of the speed of light relative to Z'. The

speed of the passenger relative to is 92.5% of the speed of light. The relation between u , v and w is shown graphically in figure XV.20.  FIGURE XV.20

If I use the notation $u \oplus v$ to mean "combining u with v ", I can write equation 15.16.6 as $u \oplus v = \frac{u + v}{1 + uv}$.

You may notice the similarity of equation 15.16.6 [3 - function identity to the hyperbolic $\tanh(4) + 4)2) = \tanh(4) + \tanh(4)2$ 15.16.8 $1 + \tanh(4) \tanh(4)2$. Thus I can represent the speed of an object by giving the value of 4), where $[3 = \tanh(4) 15.16.9 = \ln(i---) 15.16.10$ or $4) = \tanh [3 \oplus$. The factor 4) combines simply as $\oplus 20 \oplus 2 : \oplus 1 q- \oplus 2$. 15.16.11 If you did what I suggested in section 15.3 and programmed your calculator or computer to convert instantly from one relativity factor to another, you now have a quick way of adding speeds. Example. A train trundles to the right at 90% of the speed of light ($4) = 1.47222$) relative to Z , and a passenger strolls to the right at 15% of the speed of light ($4)2 = 0.15114$) relative to Z' . The speed of the passenger relative to Z is $4) = 1.62336$, or 92.5% of the speed of light. Example. FIGURE XV.22 (Sorry - there is no figure XV.21.)

36 An ocean liner ' sails serenely eastwards at a speed $3 = 0.9c$ ($\oplus = 2.29416$) relative to the ocean . A passenger ambles athwartships at a speed $32 = 0.5c$ relative to the ship. What is the velocity of the passenger relative to the ocean? The northerly component of her velocity is given by equation 15.16.5b, and is $0.21794c$. Her easterly component is just $0.9c$. Her velocity relative to the ocean is therefore $0.92601 c$ in a direction $13 \oplus 37'$ north of east. Exercise. Show that, if the speed of the ocean liner is 3 and the athwartships speed of the passenger is 32 , the resultant speed 3 of the passenger relative to the ocean is given by $= - 332 15.16.12$ and that her velocity makes an angle c with the velocity of the ship given by $\tanc = 32 \times \text{fi} - 3 15.16.13$ Example. A railway train Z' of proper length $L_0 = 100$ yards thunders past a railway station Z at such a speed that the stationmaster thinks its length is only 40 yards. (Correction: It is not a matter of what he "thinks". What I should have said is that the length of the train, referred to a reference frame Z in which the stationmaster is at rest, is 40 yards.) A dachshund waddles along the corridor towards the front of the train. (A dachshund, or badger hound, is a cylindrical dog whose proper length is normally several times its diameter.) The proper length 10 of the dachshund

is 24 inches, but to a seated passenger, it appears to be... no, sorry, I mean that its length, referred to the reference frame Z' , is 15 inches. What is the length of the dachshund referred to the reference frame Z in which the stationmaster is at rest? We are told, in effect, that the speed of the train relative to the station is given by $\gamma = 2.5$, and that the speed of the dachshund relative to the train is given by $\gamma_2 = 1.6$. So how do these two gammas combine to make the factor γ for the dachshund relative to the station? There are several ways in which you could do this problem. One is to develop a general algebraic method of combining two gamma factors. Thus: Exercise. Show that two gamma factors combine according to $\gamma^2 = \gamma_1^2 \gamma_2^2 / (1 + \gamma_1 \gamma_2 v_1 v_2 / c^2)$. 15.16.14

37 I'll leave you to try that. The other way is to take advantage of the programme you wrote when you read section 15.3, by which you can instantaneously convert one relativity factor to another. Thus you instantly convert the gammas to phis. Thus $\phi = \gamma^{-1} = 0.4$ and $\phi_2 = \gamma_2^{-1} = 0.6154$. $\phi = 1.04697$. $\phi_2 = 2.61377$. 15.16.14 gets? Therefore, referred to the railway station, the length of the dachshund is 24/7 inches. = 3.5 15.17 Aberration of Light (a) (b) FIGURE XV.23 To Apex The direction of Earth's velocity on any particular date is called the Apex of the Earth's Way. in part (a) of figure XV.23 i show Earth moving towards the apex at speed v , and light coming from a star at speed c from an angle Z from the apex. The x - and y - components of the velocity of light are respectively $-c \cos Z$ and $-c \sin Z$. Relative to Earth (part (b)), the x' - and y' -components are, by equations 15.16.2 and 15.16.3 (or rather their inverses) $c \cos Z + v$ and $c \sin Z / (1 + (v/c) \cos Z)$

38 You can verify that the orthogonal sum of these two components is c , as it should be according to our fundamental assumption that the speed of light is the same referred to all reference frames in uniform relative motion. The apparent direction of the star is therefore given by $\sin Z' = \sin Z / (1 + (v/c) \cos Z)$ 15.17.1 It is left as an exercise to show that, for small v/c , this becomes $v \sin Z / c$ 15.17.2 with $v = 29.8 \text{ km/s}$, v/c is about 10^{-8} . More details about aberration of light, including the derivation of equation 15.17.2, can be found in Celestial Mechanics, Section 11.3. 15.18 Doppler Effect It is well known that the formula for the Doppler effect in sound is different according to whether it is the source or the observer that is in

motion. An answer to the question "Why should this be?" to the effect that "Oh, that's just the way the algebra works out" is obviously unsatisfactory, so I shall try to explain why, physically, there is a difference. Then, when you have thoroughly understood that observer in motion is an entirely different situation from source in motion, and the formulas must be different, we shall look at the Doppler effect in light, and we'll return to square one when we find that the formulas for source in motion and observer in motion are the same! This section on the Doppler effect will probably be rather longer than it need be, just because some aspects interested me - but if you find it too long, just skip the parts that aren't of special interest to you. These will quite likely include the parts on the ballistic Doppler effect. First, we'll deal with the Doppler effect in sound. All speeds are supposed to be very small compared with the speed of light, so that we need not trouble ourselves with Lorentz transformations. First, let's deal with observer in motion (figure XV.24). When the source is at rest, it emits concentric equally-spaced spherical wavefronts at some frequency. When an observer moves towards the source, he will pass these wavefronts at a higher frequency than the frequency at which they were emitted, and that is the cause of the Doppler effect with a stationary source and moving observer.

39 FIGURE XV.24 Now, we'll look at the source-in-motion situation. (Figure XV.25). FIGURE XV.25 Here we see that the wavefronts are not equally spaced, but are compressed ahead of the motion of the source, and for that reason they will pass a stationary observer at a higher frequency than the frequency at which they were emitted. Thus the nature of the effect is a little different according to whether it is the source or the observer that is in motion, and thus one would not expect identical equations to describe the two situations. We shall move on shortly to discuss the effect quantitatively and develop the relevant equations. I shall assume that the reader is familiar with the usual relation connecting

40 wavelength, frequency and speed of a wave. Nevertheless I shall write down the relation in large print, three times, just to make sure: $\text{SPEED} = \text{FREQUENCY} \times \text{WAVELENGTH}$ $\text{FREQUENCY} = \text{SPEED} + \text{WAVELENGTH}$ $\text{WAVELENGTH} = \text{SPEED} + \text{FREQUENCY}$ I am going to start with the Doppler effect in sound, where the speed of the signal is constant with respect to the medium than transmits the sound - usually air. I

shall give the necessary formulas for source and observer each in motion. If you want the formulas for one or the other stationary, you just put one of the speeds equal to zero. The speeds of the source S and of the observer O relative to the air will be denoted respectively by u and v and the speed of sound in air will be denoted by c . The situation is shown in figure XV.26. 1;2

FIGURE XV.26 The relevant formulas are shown below: Source Observer

	Frequency	Speed	Wavelength
Source	ν_0	u	$\lambda_0 = c/\nu_0$
Observer	ν	v	$\lambda = c/\nu$

The way we work this table is just to follow the arrows. Starting at the top left, we suppose that the source emits a signal of frequency ν_0 . The speed of the signal relative to the source is $c - u$, and so the wavelength is $(c - u)/\nu_0$. The wavelength is the same

41 for the observer (we are supposing that all speeds are very much less than the speed of light, so the Lorentz factor is effectively 1.) The speed of sound relative to the observer is $c - v$, and so the frequency heard by the observer is the last (upper right) entry of the table. Two special cases: a. Observer in motion and approaching a stationary source at speed v . $-v$. In that case the frequency heard by the observer is $\nu_1 = 0$ and $\nu_2 = \nu = \nu_0(1 + v/c)$. 15.18.1 b. Source in motion and approaching a stationary observer at speed v . O . In that case the frequency heard by the observer is $\nu_1 = \nu$ and $\nu_2 = \nu: \nu_0/(1 - v/c) \nu_0 + (v/c) + (v/c): + \dots$. 15.18.2 Thus the formulas for source in motion and observer in motion differ in the second order of (v/c) . We might now consider reflection. Thus, suppose you approach a brick wall at speed v while whistling a note of frequency ν_0 . What will be the frequency of the echo that you hear? Let's make the question a little more general. A source S , emitting a whistle of frequency ν_0 , approaches a brick wall M at speed v . A separate observer O approaches the wall (from the same side) at speed v_2 . And, for good measure, let's have the brick wall moving at speed v_3 . (The reader may notice at this point that theoretical physics is rather easier than experimental physics.) The situation is shown in figure XV.27. $U_1 \quad c \quad S \quad \diamond \quad O \quad \diamond \quad M$ FIGURE XV.27 We construct a table similar to the previous one.

42

	Frequency	Speed	Wavelength	Source	Mirror	Mirror	Observer
before reflection	ν_0	u	$\lambda_0 = c/\nu_0$	S	M	M	O
after reflection	ν	v	$\lambda = c/\nu$	S	M	M	O

At all times, the speed relative to the air is c . The answer to our initial question, in which the source and the observer were one and the same, and the mirror (wall) was stationary is found by putting $\nu_1 = \nu_2 = v$ and $\nu_3 = 0$ in the last (top right) formula in the table. This results in

15.18.3 So much for the Doppler effect in sound. Before moving on to light, I want to look at what I shall call the Doppler effect in ballistics, or "cops and robbers". An impatient reader may safely skip this discussion of ballistic Doppler effect. A police ("cop") car is chasing a stolen car driven by robbers. The cop car is the "source" and the robber's car (or, rather the car that they have stolen, for it is not theirs) are the "observers". The cop car ("source") is travelling at speed v and the robbers ("observer") is travelling at speed v_2 . The cops are firing bullets (the "signal") towards the robbers. (No one gets hurt in this thought experiment, which is all make-believe.) The bullets leave the muzzle of the revolver at speed c (that is the speed of the bullets, and is nothing to do with light) relative to the revolver, and hence they travel (relative to the lamp-posts at the side of the road) at speed $c + v$ and relative to the robbers at speed $c + v - v_2$. The cops fire bullets at frequency ν_0 , and our task is to find the frequency with which the bullets are "received" by the robbers. The distance between the bullets is the "wavelength". This may not be a very important exercise, but it is not entirely pointless, for fairness dictates that, when we are considering (even if only to discard) possible plausible mechanisms for the propagation of light, we might consider, at least briefly, the so-called "ballistic" theory of light propagation, in which the speed of light through space is equal to the speed at which it leaves the source plus the speed of the source. Some readers may be aware of the Michelson-Morley experiment. That experiment demonstrated that light was not propagated at a speed that was constant with respect to some all-pervading "luminiferous aether" - but it must be noted that it did

43 nothing to prove or disprove the "ballistic" theory of light propagation, since it did not measure the speed of light from moving sources. In the intervening years, some attempts have indeed been made to measure the speed of light from moving sources, though their interpretation has not been free from ambiguity. I now construct a table showing the "frequency", "speed" and "wavelength" for ballistic propagation in exactly the same way as I did for sound.

Source	Observer	Frequency	ν_0	Speed	c	Wavelength
C/V_0	$C - U + U$	V_0	ν_0	C	$c + v_1 - v_2$	C/V_0

In order not to spend longer on "ballistic" propagation than is warranted by its importance, I'll just let the reader spend as much or as little time pondering over this table as he or she wishes. Just one small point might be noted, namely that the formulas for "observer in motion" and "source in motion" are the same. For completeness

rather than for any important application, I shall also construct here the table for "reflection". A source of bullets is approaching a mirror at speed v_1 . An observer is also approaching the mirror, from the same side, at speed v_2 . And the mirror is moving at speed v_3 , and reflection is elastic (the coefficient of restitution is 1.) You are free to put as many of these speeds equal to zero as you wish. The entries for "speed" give the speed relative to the source or mirror or observer. The speed relative to stationary lampposts at the side of the road is $c + v$ before reflection and $c + v - 2v_3$ after reflection.

44

Frequency	Source	Mirror before reflection	Mirror after reflection	Observer
$c + u$	$1 - \frac{v_3}{c}$	$\frac{v_0}{c + u}$	$\frac{v_0}{c + u + u_2 - 2u_3}$	$\frac{v_0}{c + u}$
Speed	$c + v_1 - v_3$	$c + v_1 + v_2 - 2v_3$	c	c
Wavelength	$\frac{c}{v_0}$	$\frac{c}{v_0}$	$\frac{c}{v_0}$	$\frac{c}{v_0}$

We now move on to the only aspect of the Doppler effect that is really relevant to this chapter, namely the Doppler effect in light. In the previous two situations I have been able to assume that all speeds were negligible compared with the speed of light, and we have not had to concern ourselves with relativistic effects. Here, however, the signal is light and is propagated at the speed of light, and this speed is the same whether referred to the reference frame in which the source is stationary or the observer is stationary. Further, the Doppler effect is noticeable only if source or observer are moving at speeds comparable to that of light. We shall see that the difference between the frequency of a signal relative to an observer and the frequency relative to the source is the result of two effects, which, while they may be treated separately, are both operative and in that sense inseparable. These two effects are the Doppler effect proper, which is a result of the changing distance between source and observer, and the relativistic dilation of time. I am going to use the symbol T to denote the time interval between passage of consecutive crests of an electromagnetic wave. I'll call this the period. This is merely the reciprocal of the frequency ν . I am going to start by considering a situation in which a source and an observer are receding from each other at a speed v . I have drawn this in figure XV.27, which is referred to a frame in which the observer is at rest. The speed of light is c . O S

FIGURE XV.27 Let us suppose that S emits an electromagnetic wave of period $T_0 = 1/\nu_0$ referred to the frame in which S is at rest. We are going to have to think about four distinct period of frequencies:

45 1. The time interval between the emission of consecutive crests by S

referred to the reference frame in which S is at rest. This is the period T_0 and the frequency ν_0 that we have just mentioned. 2. The time interval between the emission of consecutive crests by S referred to the reference frame in which O is at rest. By the relativistic formula for the dilation of time this is γT_0 or $T_0 / \sqrt{1 - v^2/c^2}$. 3. The time interval between the reception of consecutive crests by O as a result of the increasing distance between O and S (the "true" Doppler effect, as distinct from time dilation) referred to the reference frame in which S is at rest. This is $T_0 + v/c$. 4. The time interval between the reception of consecutive crests by O as a result of the increasing distance between O and S (the "true" Doppler effect, as distinct from time dilation) referred to the reference frame in which O is at rest. This is $\gamma (T_0 + v/c)$. 5. This, of course, is what O "observes", and, when you do the trivial algebra, you find that this is $T = T_0 \sqrt{1 - v^2/c^2} / (1 - v/c)$. 6. In terms of frequency, $\nu = \nu_0 \sqrt{1 - v^2/c^2} / (1 - v/c)$. If source and observer approach each other at speed v , the result is $\nu = \nu_0 \sqrt{1 - v^2/c^2} / (1 - v/c)$. 7. The factor $\sqrt{1 - v^2/c^2} / (1 - v/c)$ is often denoted by the symbol k , and indeed that was the symbol I used in section 15.3 (see equation 15.3.3). Exercise. Expand equation 15.18.9 by the binomial theorem as far as $(v/c)^2$: and compare the result with equations 15.8.1 and 15.8.2.

46 make it $\nu = \nu_0 (1 + v/c)$ + Question.  ν FIGURE XV.28 O c An observer O sends an electromagnetic signal of frequency ν_0 at speed c to a mirror that is receding at speed v . When the reflected signal arrives back at the observer, what is its frequency (to first order in v/c)? Is it $\nu_0(1 - v/c)$ or is it $\nu_0(1 - 2v/c)$? I can think offhand of two applications of this. If you examine the solar Fraunhofer spectrum reflected of the equatorial limb of a rotating planet, and you observe the fractional change $\Delta\nu/\nu_0$ in the frequency of a spectrum line, will this tell you v/c or $2v/c$, where v is the equatorial speed of the planet's surface? And if a policeman directs a radar beam at your car, does the frequency of the returning beam tell him the speed of your car, or twice its speed? You could try arguing this case in court - or, better, stick to the speed limit so there is no need to do so. The answer, by the way, is $\nu = \nu_0 (1 - 2v/c)$. Redshift. When a galaxy is moving away from us, a spectrum line of laboratory wavelength L_0 will appear to have a frequency for the observer of $L = kL_0$. The fractional increase in wavelength $L - L_0$ is generally given the symbol z , which is evidently equal to $k - 1$. (Only to L_0 first order in z is it approximately equal to z . It is important to note that the definition of z is L

- L_0 , and not v/c . L_0 A note on terminology: If a source is receding from the observer the light is observed to be shifted towards longer wavelengths, and if it is approaching the observer the light is shifted towards shorter wavelengths. Traditionally a shift to longer wavelengths is called a "redshift", and a shift towards shorter wavelengths is called a "blueshift". Note, however, that if an infrared source is approaching an observer, its light is shifted towards the red, and if an ultraviolet source is receding from an observer, its light is shifted towards the blue! Nevertheless I shall continue in this chapter to refer to shifts to longer and shorter wavelengths as redshifts and blueshifts respectively. Example. -- X R G FIGURE XV.29

47 A red galaxy R of wavelength 680.0 nm and a green galaxy G of wavelength 520.0 nm are on opposite sides of an observer X, both receding from him/her. To the observer, the wavelength of the red galaxy appears to be 820.0 nm, and the wavelength of the green galaxy appears to be 640.0 nm. What is the wavelength of the green galaxy as seen from the red galaxy? Solution. We are told that k for the red galaxy is $82/68 = 1.20588$, or $z = 0.20588$, and that k for the green galaxy k is $64/52 = 1.23077$, or $z = 0.23077$. Because of the preparation we did in section 15.3, we can instantly convert these to θ . Thus for the red galaxy $\theta = 0.187212$ and for the green galaxy $\theta = 0.207639$. The sum of these is 0.394851. We can instantly convert this to $k = 1.48416$ or $z = 0.48416$. Thus, as seen from R, the wavelength of G is 771.8 nm. Alternatively. Show that the factor k combines as $k \circ k = kk$: 15.18.11 and verify that θ : 64: 1.48416. Show also that the redshift factor z combines as $z \circ z = z + z$: 15.18.12 15.19 The Transverse and Oblique Doppler Effects I pointed out in section 15.18 that the observed Doppler effect, when the transmitted signal is electromagnetic radiation and observer or source or both are travelling at speed comparable to that of light, is a combination of two effects - the "true" Doppler effect, caused by the changing distance between source and observer, and the effect of time dilation. This raises the following question: If a source of light is moving at right angles to (transverse to) the line joining observer to source, will the observer see a change in frequency or wavelength, even though the distance between observer and source at that instant is not changing? The answer is yes, certainly, and the effect is sometimes called the "transverse Doppler effect", although it is the effect of relativistic time dilation rather than of a true Doppler effect. Thus let us suppose that a source is moving transverse to the

line of sight at a speed described by its parameter β or β , and that the period of the radiation referred to the reference frame in which the source is at rest is T_0 and the frequency is ν_0 . The time interval between emission of consecutive wavecrests when referred to the frame in which the observer is at rest is longer by the gamma- factor, and the frequency is correspondingly less. That is, the frequency, referred to the observer's reference frame, is $\nu = \nu_0 / \gamma = \nu_0 (1 - \beta^2)^{-1/2}$. 15.19.1

48 The light from the source is therefore seen by the observer to be redshifted, even though there is no radial velocity component. This raises a further question. Suppose a source is moving almost but not quite at right angles to the line of sight, so that it has a large transverse velocity component, and a small velocity component towards the observer. In that case, its "redshift" resulting from the time dilation might be appreciable, while its "blueshift" resulting from "true" Doppler effect (the decreasing distance between source and observer) is still very small. Therefore, even though the distance between source and observer is slightly decreasing, there is a net redshift of the spectrum. This is in fact correct, and is the "oblique Doppler effect". In figure XV.30, a source S is moving at speed β times the speed of light in a direction that makes an angle θ with the line of sight. It is emitting a signal of frequency ν_0 in S. (I am here using the frame "in S" as earlier in the chapter to mean "referred to a reference frame in which S is at rest.") The signal arrives at the observer O at a slightly greater frequency as a result of the decreasing distance of S from O, and at a slightly lesser frequency as a result of the time dilation, the two effects opposing each other. S FIGURE XV.30 $\nu_0 \gamma (1 - \beta \cos \theta)$ The frequency of the received signal at O, in O, is $\nu = \nu_0 \gamma (1 - \beta \cos \theta)$ For a given angle θ the redshift is zero for a speed of $2 \cos \theta / (1 + \cos^2 \theta)$ 15.19.2 15.19.3

49 or, for a given speed, the direction of motion resulting in a zero redshift is given by $\beta \cos^2 \theta - 2 \cos \theta + \beta = 0$. 15.19.4 This relation is shown in figure XV.30. (Although equation 15.19.4 is quadratic in $\cos \theta$, there is only one real solution θ for β between 0 and 1. Prove this assertion.) It might be noted that if the speed of the source is 99.99% of the speed of light the observer will see a redshift unless the direction of motion of S is no further than $9^\circ 36'$ from the line from S to O. That is worth repeating: S is moving very close to the speed of light in a direction that is close to being directly towards the

observer; the observer will see a redshift. Equation 15.19.2, which gives v as a function of θ for a given β , will readily be recognized at the equation of an ellipse of eccentricity β , semi minor axis v_0 and semi major axis γv_0 . This relation is shown in figure XV.31 for several β . The curves are red where there is a redshift and blue where there is a blueshift. There is no redshift or blueshift for $\beta = 0$, and the ellipse for that case is a circle and is drawn in black. β 0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2 0.1 0 0 versus θ for zero redshift. I I I I I I I I 10 20 30 40 50 60 70 80 90 0 An alternative and perhaps more useful way of looking at equation 15.19.2 is to regard it as an equation that gives β as a function of θ for a given Doppler ratio v/v_0 . For example, if the Doppler ratio of a galaxy is observed to be 0.75, the velocity vector of the galaxy could be any

50 arrow starting at the black dot and ending on the curve marked 0.75. The curves are ellipses with semi major axis equal to $1/\sqrt{1 - (v/v_0)^2}$ and semi minor axis $1/\gamma - (v/v_0)^2$. β 0.2 0.4 0.6 0.8 FIGURE XV.31

51 $V/V_0 - 0.00 0.25 0.50$ FIGURE XV.32 15.20 Acceleration $Y \times X$! FIGURE XV.33

52 Figure XV.33 shows two reference frames, Z and Z' , the latter moving at speed 1 ; with respect to the former. A particle is moving with acceleration a' in Z' . ("in Z' " = "referred to the reference frame Z' ".) The velocity is not necessarily, of course, in the same direction as the acceleration, and we'll suppose that its velocity in Z' is u' . The acceleration and velocity components in Z' are a'_x, a'_y, u'_x, u'_y . What is the acceleration of the particle in Z ? We shall start with the x -component. The x -component of its acceleration in Z is given by du_x/dt where $U_x = u'_x + 1$; 15.16.2 $1 + u'_x/c$ and $t = \gamma(t' + vu'_x/c^2)$. 15.5.19 Equations 15.16.2 and 15.5.19 give us $du_x = du'_x \gamma(1 + U'_x V/C^2)^2 C^3 t^{1,2}$, and $dt = \gamma dt' + \gamma v dx'/c$. 15.20.3 On substitution of these into equation 15.20.1 and a very little algebra, we obtain $du_x = \gamma^2 (1 - U'_x^2/C^2)^{3/2} du'_x$. 15.20.4 The y -component of its acceleration in Z is given by du_y/dt . 15.20.5 We have already worked out the denominator dt (equation 15.20.3). We know that $dy/dt = \gamma(1 + U'_x V/C^2)$. 15.16.3

53 from which $du_y = \gamma^2 (1 - U'^2/C^2)^{3/2} du'_y$. 15.20.6 Divide equation 15.20.6 by equation 15.20.3 to obtain $1 + VU'_x/C^2$

duv,. 15.20.6 $ay = c^2(l+VU_{x,}/C^2)^3a_{x,} + (l+VU_{x,}/C^2)^2a_{x,}$. 15.20.7 21.

Mass It is well known that "in relativity" the mass of an object increases as its speed increases. This may be well known, but I am not certain that it is a very precise statement of the true situation. Or at least it is no more precise than to say that the length of a rod decreases as its speed increases. The length of a rod when referred to a frame in which it is at rest is called its proper length l_0 , and the mass of a body when referred to a frame in which it is at rest is called its rest mass m_0 , and both of these things are invariant. The length of a rod when referred to a reference frame that is moving with respect to it (i.e., in Minkowski language, its component along an inclined axis) and the mass of a body referred to a frame that is moving with respect to it may indeed be different from the proper length of the rod or the rest mass of the body. In order to derive the FitzGerald-Lorentz contraction, we had to think about what we mean by "length" and how to measure it. Likewise, in order to derive the "relativistic increase of mass" (which may be a misnomer) we have to think about what we mean by mass and how to measure it. The fundamental unit of mass used at present in science is the International Prototype Kilogram, a platinum-iridium alloy, held in Savres, Parris, France. On order to determine the mass, or inertia, of another body, we need to carry out an experiment to compare its reluctance to accelerate when a force is applied to it with the reluctance of the standard kilogram when the same force is applied. We might, for example, attach the body to a spring, stretch the spring, let go, and see how fast the body accelerates. Then we carry out the same experiment with the International Prototype Kilogram. Or we might apply an impulse (Ildt - see Chapter 8) to the body and to the Kilogram, and measure the speed immediately after applying the impulse. This might be done, for example, but striking the body and the Kilogram with a golf club, or, for a more controlled experiment, one could press each body up against a compressed spring, release the spring, and measure the resulting speed imparted to the body and to the Kilogram. (It is probable that the International Prototype Kilogram) is kept under some sort of guard, and its

54 curators may not altogether appreciate such experiments, so perhaps these experiment has better remain Thought Experiments.) Yet another method would be to cause the body and the Kilogram to collide with each other, and to assume that the collision is elastic (no internal degrees of freedom) and that momentum (defined as the product of mass and velocity) are conserved.

All of these experiments measure the reluctance to accelerate under a force; in other words the inertia or the inertial mass or just the mass of the body. Another possible experiment to determine the mass of the body would be to place it and the Kilogram at a measured distance from another mass (such as the Earth) and measure the gravitational force (weight) of each. One has an uneasy feeling that this sort of measurement is somehow a little different from the others, in that it isn't a measure of inertia. Some indeed would differentiate between the inertial mass and the gravitational mass of a body, although the two are in fact observed to be strictly proportional to one another. Some would not find the proportionality between inertial and gravitational mass particularly remarkable; to others, the proportionality is a surprising fact of the profoundest significance. In this chapter we do not deal with general relativity or with gravity, so we shall think of mass in terms of its inertia. I am going to measure the ratio of two masses (one of which might be the International Prototype Kilogram) by allowing them to collide, and their masses are to be defined by assuming that the momentum of the system is conserved in all uniformly moving reference frames. Figure XV.34 shows two reference frames, Z and Z' , the latter moving to the right at speed v relative to the former. Two bodies, of identical masses in Z' (i.e. referred to the frame Z'), are moving at speed u' in Z' , one of them to the right, the other to the left. Their mutual centre of mass is stationary in Z' . FIGURE XV.34 X ! Now let us refer the situation to the frame Z (see figure XV.35).

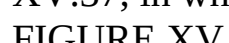
55 Y Z //1 //2 A x FIGURE XV.35 The total momentum of the system in Z is $m_1u_1 + m_2u_2$. But the centre of mass (which is stationary in Z') is moving to the right in Z with speed v . Therefore the momentum is also $(m_1 + m_2)U$. If they stick together upon collision, we are left with a single particle of mass $m_1 + m_2$ moving at speed v , and, because there are no external forces, the momentum is conserved. In any case, whether the collision is elastic or not, we have $m_1u_1 + m_2u_2 = (m_1 + m_2)U$. 15.21.1 $U' + U = -U' + U$ But $u = 1 + u'v/c$ and $u = 1 - u'v/c$: 15.21.2a,b Our aim is to try to find a relation between the masses and speeds referred to Z . Therefore we must eliminate v and u' from equations 15.21.1, 15.21.2a and 15.21.2b. This can be a bit fiddly, but the algebra is straightforward, and I leave it to the reader to show that the result is $m = m_0 / \sqrt{1 - u^2/c^2}$ 15.21.3 This tells us that the mass m of a body referred to Z is proportional to $1/\sqrt{1 - u^2/c^2}$, where u is its speed referred to Z . If we call the proportionality constant m_0 , then

$m = m_0 / \sqrt{1 - u^2/c^2}$ If $u = 0$, then $m = m_0$, and m_0 is called the rest mass, and it is the mass when referred to a frame in which the body is at rest. The mass m is generally called the relativistic mass, and it is the mass when referred to a frame in which the speed of the body is u . Equation 15.21.4 gives the mass referred to Z assuming that the mass is at rest in Z' . But what if the mass is not at rest in $Y Z$ $\diamond$ t ; X FIGURE XV.36 In figure XV.36 we see a mass m' moving with velocity u' in Z' . Referred to Z its mass will be m , where $m = \frac{m'}{\sqrt{1 - u'^2/c^2}}$ 15.21.5 Its velocity u will be in a different direction (referred to Z) from the direction of u' in Z' , and the speed will be given by $u^2 = u_x^2 + u_y^2$, 15.21.6 where u_x and u_y are given by equations 15.16.2 and 15.16.3. Substitute equations 15.21.6, 15.16.2 and 15.16.3 into equation 15.21.5. The objective is to replace u entirely by primed quantities. The algebra is slightly boring, but it is worth persisting. You will find that u'^2 appears when you use equation 15.16.3. Replace that by $u'^2 - u'^2 \cos^2 \theta$ Also 3, $x \diamond$ write $1/(1 - v^2/c^2)$ for $\diamond^2$. After a little while you should arrive at

57 $m = m' \gamma (1 + \frac{v u'_x}{c^2})$ 15.21.7 The transformation for mass between the two frames depends only on the x' component of its velocity in Z' . It would have made no difference, other than to increase the tedium of the algebra, if I had added $+u_z$ to the right hand side of equation 15.21.6. The inverse of equation 15.21.7 is found in the usual way by interchanging the primed and unprimed quantities and changing the sign of v : $m' = \gamma (m - \frac{v u_x}{c^2})$ 15.21.8 Example. Let's return to the problem of the dachshund that we met in section 15.16. A railway train Z' is trundling along at a speed $v/c = 0.9$ ($\gamma = 2.294$). The dachshund is waddling towards the front of the train at a speed $u'_x/c = 0.8$. In the reference frame of the train Z' the mass of the dog is $m' = 8$ kg. In the reference frame of the railway station, the mass of the dog is given by equation 15.21.7 and is 31.6 kg. (Its length is also much compressed, so it is very dense when referred to Z and is disc-shaped.) I leave it to the reader to show that the rest mass of the dog is 4.8 kg. 15.22 Momentum The linear momentum p of a body, referred to a frame Z , is defined as $p = mu$. 15.22.1 Here m and u are its mass and velocity referred to Z . Note that m is not the rest mass. Example. The rest mass of a proton is 1.67×10^{-27} kg. What is its momentum referred to a frame in which it is moving at 99% of the speed of light? Answer = 3.51×10^{-19} kg m/s 15.23 Some Mathematical Results Before proceeding with the next section, I just want to establish few

mathematical results, so that we don't get bogged down in heavy algebra later on when we should be concentrating on understanding physics. First, if $\gamma = (1 - u^2/c^2)^{-1/2}$, 15.23.1

58 Then, by trivial differentiation, $du/dt = u^2/c^2$ 15.23.2 From this, we quickly find that $\gamma^2 = 1/(1 - u^2/c^2)$ Now for a small result concerning a scalar (dot) product. Let $\mathbf{A}$ be a vector such that $\mathbf{A} \cdot \mathbf{A} = A^2$. Then $d(A^2)/dt = 2\mathbf{A} \cdot d\mathbf{A}/dt$ 15.23.3 15.23.5 ... $\mathbf{A} \cdot d\mathbf{A}/dt = A^2/t$ 15.23.6 We can now safely proceed to the next section. 15.24 Kinetic Energy If a force $\mathbf{F}$ acts on a particle moving with velocity $\mathbf{u}$, the rate of doing work - i.e. the rate of increase of kinetic energy T is $1/\gamma = \mathbf{F} \cdot \mathbf{u}$. But $\mathbf{F} = d\mathbf{p}/dt$, where $\mathbf{p} = \gamma m_0 \mathbf{u}$. (A point about notation may be in order here. I have been using the symbol $\mathbf{v}$ and t ; for the velocity and speed of a frame Z' relative to a frame Z , and my choice of axes without significant loss of generality has been such that $\mathbf{v}$ has been directed parallel to the x -axis. I have been using the symbol u for the velocity (speed = u) of a particle relative to the frame Z . Usually the symbol γ has meant $(1 - v^2/c^2)^{-1/2}$, but here I am using it to mean $(1 - u^2/c^2)^{-1/2}$. I hope that this does not cause too much confusion and that the context will make it clear. I toyed with the idea of using a different symbol, but I thought that this might make matters worse. Just be on your guard, anyway.) We have, then $\mathbf{F} = m_0 d(\gamma \mathbf{u})/dt$ 15.24.1 and therefore $1/\gamma = m_0 (u^2 + \gamma^2 u \cdot du/dt)$. 15.24.2

59 Making use of equations 15.23.5 and 15.23.6 we obtain $\gamma = \gamma_0 \cosh \theta$. 15.24.3 Integrate with respect to time, with the condition that when $\theta = 0$, $T = 0$, and we obtain the following expression for the kinetic energy: $T = (\gamma - 1)m_0 c^2$. 15.24.4 Exercise. Expand γ by the binomial theorem as far as u^2/c^2 , and show that, to this order, $T = \frac{1}{2} m_0 u^2$. I here introduce the dimensionless symbol $T_K = T/m_0 c^2$ 15.24.5 to mean the kinetic energy in units of $m_0 c^2$. The second half of this was already given as equation 15.3.5. 15.25 Addition of Kinetic Energies I want now to consider two particles moving at nonrelativistic speeds - by which I mean $1/2$ that the kinetic energy is given to a sufficient approximation by the expression $\frac{1}{2} m u^2$ and so that parallel velocities add linearly. Consider the particles in figure XV.37, in which the velocities are shown relative to laboratory space.  FIGURE XV.37 Referred to laboratory space, the kinetic energy is $\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$. However, the centre of mass is moving to the right with

speed // -- $(m_1 u_1 + m_2 u_2)/(m_1 + m_2)$, and, referred to centre of mass space, the kinetic energy is $\frac{1}{2} m_1 (u_1 - u)^2 + \frac{1}{2} m_2 (u_2 - u)^2$. On the other hand, if we refer the situation to a frame in which m_1 is at rest, the kinetic energy is $\frac{1}{2} m_2 (u_2 + u)^2$, and, if we refer the situation to a frame in which m_2 is at rest, the kinetic energy is $\frac{1}{2} m_1 (u_1 + u)^2$.

60 All we are saying is that the kinetic energy depends on the frame to which speed are referred - and this is not something that crops up only for relativistic speeds. Let us put some numbers in. Let us suppose, for example that $m_1 = 3\text{kg}$, $u_1 = 4\text{ms}^{-1}$, $m_2 = 2\text{kg}$, $u_2 = 4\text{ms}^{-1}$, $u = 1.2\text{ms}^{-1}$. In that case, the kinetic energy referred to laboratory space is referred to centre of mass space is referred to m_1 is referred to m_2 is 33 J, 29.4 J, 49 J, 73.5 J. In this case the kinetic energy is least when referred to centre of mass space, and is greatest when referred to the lesser mass. Exercise. Is this always so, whatever the values of m_1 , m_2 , u_1 and u_2 ? It may be worthwhile to look at the special case in which the two masses are equal (m) and the two speeds (whether in laboratory or centre of mass space) are equal (u). In that case the kinetic energy in laboratory or centre of mass space is $\frac{1}{2} m u^2$, while referred to either of the masses it is $m u^2$. We shall now look at the same problem for particles travelling at relativistic speeds, and we shall see that the kinetic energy referred to a frame in which one of the particles is at rest is very much greater than (not merely twice) the energy referred to a centre of mass frame. If two particles are moving towards each other with "speeds" given by β and β' in centre of mass space, the β of one relative to the other is given by equation 15.16.14, and, since $K = \frac{1}{2} m_0 \gamma^2 - 1$, it follows that if the two particles have kinetic energies K and K' in centre of mass space (in units of the $m_0 c^2$ of each), then the kinetic energy of one relative to the other is $K : K' : K + K' + K K' : (K + 2)(K' + 2)$. 15.25.1 If two identical particles, each of kinetic energy K times $m_0 c^2$ approach each other, the kinetic energy of one relative to the other is $K = 2K(K+2)$. 15.25.2

61 For nonrelativistic speeds as $K \rightarrow 0$, this tends to $K = 4K$, as expected. Let us suppose that two protons are approaching each other at 99% of the speed of light in centre of mass space ($K = 6.08881$). Referred to a frame in which one proton is at rest, the kinetic energy of the other will be $K = 98.5025$, the relative speeds being 0.99995 times the speed of light. Thus $K = 16K$, rather than merely $4K$ as in the nonrelativistic calculation. For more energetic

particles, the ratio K/K is even more. These calculations are greatly facilitated if you wrote, as suggested in section 15.3, a program that instantly connects all the relativity factors given there. Exercise. Two protons approach each other, each having a kinetic energy of 500 GeV in laboratory or centre of mass space. (Since the two rest masses are equal, these two spaces are identical.) What is the kinetic energy of one proton in a frame in which the other is at rest? (Answer: I make it 535 TeV.) The factor K (the kinetic energy in units of $m_0 c^2$) is the last of several factors used in this chapter to describe the speed at which a particle is moving, and I take the opportunity here of summarising the formulas that have been derived in the chapter for combining these several measures of speed. These are 15.16.7 15.16.14 15.18.11 $z \circ z^2 = z, z^2 + z + z^2$. 15.18.12 15.25.1 $02002: + 02$. 15.16.11 If the two speeds to be combined are equal, these become $\circ, \circ = 2$, 15.25.3 $\diamond, \circ \diamond = 2 \diamond - 1$ 15.25.4 $k \circ k = k^2$ 15.25.5 $z, \circ z, z, (z, + 2)$ 15.25.6 $K, \circ K, = 2K, (K, + 2)$. 15.25.7

62 2 $\diamond$. 15.25.8 These formulas are useful, but for numerical examples, if you already have a program for interconverting between all of these factors, the easiest and quickest way of combining two "speeds" is to convert them to $\circ$. We have seen examples of how this works in sections 15.16 and 15.18. We can do the same thing with our example from the present section when combining two kinetic energies. Thus we were combining two kinetic energies in laboratory space, each of magnitude $K = 6.08881$ ($\circ = 2.64665$). From this, $\circ = 5.29330$, which corresponds to $K = 98.5025$. 15.26 Energy and mass The nonrelativistic expression for kinetic energy $T = \frac{1}{2} \mu v^2$ has just one term in it, a term which depends on the speed. The relativistic expression which approximates to the nonrelativistic expression at low speeds) can be written $T = mc^2 - m_0 c^2$ that is, a speed-dependent term minus a constant term. The kinetic energy can be thought of as the excess over the energy over the constant term $m_0 c^2$. The expression $m_0 c^2$ is known as the rest-mass energy. The sum of the kinetic energy and the rest-mass energy is the "total energy", or just the "energy" E : $E = T + m_0 c^2 = mc^2$ 15.26.1 This means that, if the kinetic energy of a particle is zero, the total energy of the particle is not zero - it still has its rest-mass energy $m_0 c^2$. Of course, giving the name "rest-mass energy" to the constant term $m_0 c^2$, and calling the speed-dependent term mc^2 the "total energy" and writing the famous equation $E = mc^2$, does not by itself immediately and directly tell us that "matter" can be converted to

"energy" or the other way round. Whether such conversion can in fact take place is a matter for experiment and observation to determine. The equation by itself merely tells us how much mass is held by a given quantity of energy, or how much energy is held by a given quantity of mass. That entities that we traditionally think of as "matter" can be converted into entities that we traditionally think of as "energy" is well established with, for example, the "annihilation" of an electron and a positron ("matter" and "antimatter") to form photons ("energy") as is the inverse process of pair production (production of an electron-positron pair from a gamma ray in the presence of a third body). 2 It is unfortunate that the main (almost the only) example of application of the equation $E = mc^2$ persistently presented to the nonscientific public is the atom bomb, whose operation actually has nothing at all to do with the equation $E = mc^2$, nor, contrary to the popular mind, is any "matter" converted to energy. I have heard it said that you can find out on the Web how to build an atom bomb, so here goes - here is how an atom bomb works. A uranium-235 nucleus is held together by strong attractive

forces between the nucleons, which, at short femtometre ranges are much stronger than the Coulomb repulsive forces between the protons. When the nucleus absorbs an additional neutron, the resulting ^{236}U nucleus is unstable and breaks up into two intermediate-mass nuclei plus two or three neutrons. The two intermediate-mass nuclei are generally not of exactly equal mass; one is usually a bit less than half of the uranium nucleus and the other a bit more than half, but that's a detail. The potential energy required to bind the nucleons together in the uranium nucleus is rather greater than the binding energy of the two resulting intermediate-mass nuclei; the difference is of order 200 MeV, and that potential energy is converted into kinetic energy of the two resulting nuclei and, to a lesser extent, the two or three neutrons released. That is all. It is merely the familiar conversion of potential binding energy (admittedly a great deal of energy) into kinetic energy. No matter, no protons, no neutrons, are "destroyed" or "converted into energy", and $E = mc^2$ simply doesn't enter into it anywhere! The rest-mass energy of a proton or a neutron is about 1 GeV, and that much energy would be released if a proton were miraculously and for no cause converted into energy. Let us hope that no one invents a bomb that will do that - though we may rest assured that that is rather unlikely. Where the equation $E = mc^2$ does come in is in the familiar observation that the mass of any nucleus other than hydrogen is a

little less than the sum of the masses of the constituent nucleons. It is for that reason that nuclear masses, even for pure isotopes, are not integral. The mass of a nucleus is equal to the sum of the masses of the constituent nuclei plus the mass of the binding energy, the latter being a negative quantity since the inter-nucleon forces are attractive forces. The equation $E = mc^2$ tells us that energy (such as, for example, the binding energy between nucleons) has mass.

15.27 Energy and Momentum A moving particle has energy arising from its momentum and also from its rest mass, and we need to find an expression relating energy to rest mass and momentum. It is fairly easy and it goes like this: $E^2 = m^2c^4 = c^2(m^2c^2 + p^2)$, $E^2 = m_0^2c^4 + p^2c^2$. Thus we obtain for the energy in terms of rest mass and momentum $E^2 = (m_0c^2)^2 + (pc)^2$.

15.27.1 If the speed (and hence momentum) is zero, the energy is merely m_0c^2 . If the rest mass is zero (as, for example, a photon) and the energy is not zero, then $E = pc = m_0c^2$. But also $E = mc^2$, so that, if the rest mass of a particle is zero and the energy is not, the particle must be moving at the speed of light. This could be regarded as the reason why photons, which have zero rest mass, travel at the speed of light. If neutrinos have zero rest mass, they, too, will travel at the speed of light; if they are not massless, they won't.

64 In addition to equation 15.27.1, which relates the energy to the magnitude of the momentum, it will be of interest to see how the components of momentum transform between reference frames. As usual, we are considering frame Z' to be moving with respect to Z at a speed v with respect to Z . There is no difficulty with the y - and z - components. We have merely $p_y = p'_y$ and $p_z = p'_z$. However: $p_x = \gamma m_0 u_x$, $m = \gamma m_0$ and $p'_x = \gamma' m'_0 u'_x$, $\gamma = 1/\sqrt{1 - v^2/c^2}$, $u_x = dx/dt$, $u'_x = dx'/dt'$ from which $1 - v^2/c^2 = (1 - u_x^2/c^2)/(1 - u'^2/c^2)$. After a little algebra, we obtain $p_x = \gamma m_0 (u_x - v) (1 - u'^2/c^2)^{1/2} (1 - u^2/c^2)^{-1/2}$. And this is $p_x - \gamma' E'/c^2 = \gamma (p'_x - \gamma' E'/c^2)$.

15.27.2 (1 - The inverse is found in the usual way: $p_x = \gamma (p'_x + \gamma' E'/c^2)$ 15.27.3 If we eliminate p'_x , from equations 15.27.2 and 15.27.3, we'll find E' in terms of E and p_x : $E' = \gamma (E - vp_x)$.

15.27.4 Thus the transformations between energy and the three spatial components of momentum is similar to the transformation between time and the three space coordinates, and are described by a similar 4-vector:

65 $\begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ 15.27.5 The reader should multiply this out to

verify that it does reproduce equations 15.27.3 and 15.27.4. 15.28 Units

It is customary in the field of particle physics to express energy (whether total, kinetic or rest-mass energy) in electron volts (eV) or in keV, MeV, GeV or TeV (10^3 , 10^6 , 10^9 , or 10^{12} eV respectively). A electron volt is the kinetic energy gained by an electron if it is accelerated through an electrical potential of 1 volt; alternatively it is the work required to move an electron through one volt. Either way, since the charge on an electron is 1.602×10^{-19} C, $1\text{ eV} = 1.602 \times 10^{-19}$ J. The use of such a unit may understandably dismay those who would insist always on expressing any physical quantity in SI units, and I am much in sympathy with this view. Yet, to those who deal daily with particles whose charge is equal to or is a small multiple or rational fraction of the electronic charge, the eV has its attractions. If you accelerate a particle through so many volts, you don't have to remember the exact value of the electronic charge or carry out a long multiplication every time you do so. One might also think of a hypothetical question such as: An electron is accelerated through 3426.7189628471 volts. What is its gain in kinetic energy? You cannot answer this in joules unless you know the value of the electronic charge to a comparable precision; but of course you do know the answer in eV. One situation that does require care is this. An α -particle is accelerated through 1000 V. What is the gain in kinetic energy? Because the charge on an α -particle is twice that of an electron, the answer is 2000 eV. Very often you know the energy of a particle (because you have accelerated it through so many volts) and you want to know its momentum; or you know its momentum (because you have measured the curvature of its path in a magnetic field) and you want to know its energy. Thus you will frequent occasion to make use of equation 15.27.1: $E^2 = (mc^2)^2 + (pc)^2$. You have to be careful to remember how many c s there are, and what is the exact value of c . Particle physicists prefer to make life easier for themselves (not necessarily for the rest of us!) by preferring not to state what the momentum of a particle is, or its rest mass,

66 but rather to give the values of pc or of mc^2 - and to express E , pc and mc^2 all in eV (or keV, MeV or GeV). Thus one may hear that $pc = 6.2$ GeV, $2mc^2 = 0.938$ GeV. More often this is expressed, somewhat idiosyncratically and in somewhat doubtful use of English, as $p = 6.2$ GeV/ c , $m_0 = 0.938$ GeV/ c^2 or in informal casual conversation (one hopes not for publication) merely as $p = 6.2$ GeV, $m_0 = 0.938$ GeV. While this may puzzle some and

raise the ire of others, it is not entirely without merit, because, provided one uses these units, the relation between energy, momentum and rest mass is then simply $E^2 = m^2 c^4 + p^2 c^2$. The practice is not confined to energy, momentum and rest mass. For example, the SI unit of magnetic dipole moment is N m T^{-1} (newton metre per tesla). Now N m (unit of torque) is not quite the same as a joule (unit of energy), although dimensionally similar. Yet it is common practice to express the magnetic moments of subatomic particles in eV T^{-1} . Thus the Bohr magneton is a unit of magnetic dipole moment equal to $9.27 \times 10^{-24} \text{ N m T}^{-1}$, and this maybe expressed as $5.77 \times 10^{-5} \text{ eV T}^{-1}$. One small detail to be on guard for is this. One may hear talk of "a 500 MeV proton". Does this mean that the kinetic energy is 500 MeV or that its total energy is 500 MeV? In this case the answer is fairly clear (although it would have been completely clear if the speaker had been explicit). The rest-mass energy of a proton is 938 MeV, so he must have been referring to the kinetic energy. If, however, he had said "a 3 GeV proton", there would be no way of deducing whether he was referring to the kinetic or the total energy. And if he had said "a 3 GeV particle", there would be no way of telling whether he was referring to its total energy, its kinetic energy or its rest-mass energy. It is incumbent on all of us - or at least those of us who wish to be understood by others - always to make ourselves explicitly clear and not to suppose that others will correctly guess what we mean.

67 15.29 Force Force is defined as rate of change of momentum, and we wish to find the transformation between forces referred to frames in uniform relative motion such that this relation holds on all such frames. Suppose that, in Z' , a mass has instantaneous mass m' and velocity whose instantaneous components are u'_x , and u'_y . If a force acts on it, then the velocity and hence also the mass are functions of time. The x-component of the force is given by $d(m' u'_x)/dt'$. 15.29.1 We want to express everything on the right hand side in terms of unprimed quantities. Thus from equation 15.21.8 and the inverse of equation 15.16.2, we obtain $m' u'_x = m \gamma (u_x - v)$. 15.29.2 Also $dt/dt' = \gamma$. 15.29.3 $dt'/dt = 1/\gamma$ Let us first evaluate $d(m \gamma u_x - m \gamma v)/dt$. In this expression, v and γ are independent of time (the frame Z' is moving at constant velocity relative to Z), and $d(m \gamma u_x)/dt$ is the x-component of the force in Z , that is F_x . Thus $d(m \gamma u_x - m \gamma v)/dt = F_x - v dm \gamma/dt$. 15.29.4 Now we need to evaluate $dm \gamma/dt$ in terms of unprimed quantities. If we start with $dt (x' J t' + - x' 15.29.5$ we'll just get more primed quantities. What we'll do

instead is to start with $dt' = \gamma(dt - \frac{v}{c^2} dx)$ (15.29.6)

68 and we'll evaluate $\frac{d}{dt}$, which, being a total derivative, is the reciprocal of $\frac{dt}{dx}$ partial derivatives are given by equations 15.15.3k and 1, while $\frac{d}{dt} = \gamma \frac{d}{dx}$.
 dt obtain $\frac{d}{dt}$. The $\frac{d}{dt}$ Hence we Thus we arrive at $\frac{d}{dt} = \gamma \frac{d}{dx}$ (15.29.7) $F' = F_x - \frac{v}{c^2} \frac{dm}{dt}$
 15.29.8 $\gamma^2 (1 - \frac{UxV}{c^2})$ The mass is not constant (i.e. $\frac{dm}{dt}$ is not zero) because there is a force acting on the body, and we have to relate the term $\frac{dm}{dt}$ to the force. At some instant when the force and velocity (in E) are F and u , the rate at which F is doing work on the body is $F \cdot u = F_x u_x + F_y u_y + F_z u_z$ and this is equal to the rate of increase of energy of the body, which is $\frac{dE}{dt}$. (In section 15.24, in deriving the expression for kinetic energy, I wrote that the rate of doing work was equal to the rate of increase of kinetic energy. Now I have just written that it is equal to the rate of increase of (total) energy. Which is right?) $\frac{dm}{dt} = \frac{1}{c^2} \frac{dE}{dt} = \frac{1}{c^2} (F_x u_x + F_y u_y + F_z u_z)$. 15.29.9 Substitute this into equation 15.29.8 and, after a very little more algebra, we finally obtain the transformation for $F'_x = \gamma (F_x - \frac{v}{c^2} \frac{dE}{dt})$ 15.29.10 The y' - and z' - components are a little easier, and I leave it as an exercise to show that $F'_y = \gamma F_y$ 15.29.11 $\gamma (1 - \frac{UxV}{c^2})$ 15.29.12 $\gamma (1 - \frac{UxV}{c^2})$ As usual, the inverse transformations are found by interchanging the primed and unprimed quantities and changing the sign of v .

69 The force on a particle and its resultant acceleration are not in general in the same direction, because the mass is not constant. (Newton's second law is not $F = ma$; it is $F = \frac{dp}{dt}$. Thus $F = \frac{d}{dt}(mu)$: $ma + \frac{d}{dt}m u$. 15.29.13 $m = m_0 \gamma$ Here $m = m_0 \gamma$ 15.29.14 $(1 - \frac{u^2}{c^2})^{-1/2}$ and so $\frac{d}{dt}m u = m_0 \frac{d}{dt}(\gamma u)$ 15.29.15 $(1 - \frac{u^2}{c^2})^{-3/2} m_0 u \frac{du}{dt}$ Thus $F = \frac{d}{dt}(mu) = \frac{d}{dt}(m_0 \gamma u) = m_0 \gamma^3 \frac{du}{dt}$ 15.29.16 $\gamma^3 = \frac{1}{(1 - \frac{u^2}{c^2})^{3/2}}$ 15.30 Electromagnetism These notes are intended to cover only mechanics, and therefore I resist the temptation to cover here special relativity and electromagnetism. I point out only that in many ways this misses many of the most exciting parts of special relativity, and indeed it was some puzzles with electromagnetism that led Einstein to formulate the theory of special relativity. One proceeds as we have done with mechanical quantities; that is, we have to define carefully what is meant by each quantity and how in principle it is possible to measure it, and then see how it transforms between frames in such a manner that the laws of physics - in particular Maxwell's equations - are the same in each. One such transformation that is found, for example, is $E' = \gamma(E + u \times B)$, so that what appears in one frame as an electric field appears in another at least

in part as a magnetic field. The Coulomb force transforms to a Lorentz force; Coulomb's law transforms to Ampere's law. Although I do no more than mention this topic here, I owe it to the reader to say just a little bit more about the speedometer that I designed in section 15.4. It is indeed true that, as the train moves forward, the net repulsive force between the two rods does diminish, although not quite as I have indicated, for one has to make the correct transformations between frames for force, current, electric field, magnetic field, and so on. But it turns out that the weights of the rods - i.e. the downward forces on them - also diminish in exactly the same ratio, and the angle between the strings remains stubbornly the same. Our trip to the patent office will be in vain. The speedometer will not work, and it remains impossible to determine the absolute motion of the train.

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CHAPTER 16 HYDROSTATICS 1. Introduction This relatively short chapter deals with the pressure under the surface of an incompressible fluid, which in practice means a liquid, which, compared with a gas, is nearly, if not quite, incompressible. It also deals with Archimedes' principle and the equilibrium of floating bodies. The chapter is perhaps a little less demanding than some of the other chapters, though it will assume a familiarity with the concepts of centroids and radius of gyration, which are dealt with in chapters 1 and 2. 2. Density There is little to be said about density other than to define it as mass per unit volume. However, this expression does not literally mean the mass of a cubic metre, for after all a cubic metre is a large volume, and the density may well vary from point to point throughout the volume. Density is an intensive quantity in the thermodynamical sense, and is defined at every point. A more exact definition of density, for which I shall usually use the symbol ρ , is $\rho = \lim_{V \rightarrow 0} \frac{m}{V}$. 16.2.1 The awful term "specific gravity" was formerly used, and is still regrettably often heard, as either a synonym for density, or the dimensionless ratio of the density of a substance to the density of water. It should be avoided. The only concession I shall make is that I shall use the symbol s to mean the ratio of the density of a body to the density of a fluid in which it may be immersed or in which it may be floating. The density of water varies with temperature, but is approximately 1 g cm^{-3} or 1000 kg m^{-3} . 3. Pressure Pressure is force per unit area, or, more precisely, $P = \lim_{A \rightarrow 0} \frac{F}{A}$. 16.3.1 There is no particular direction associated with pressure - it

acts in all directions - and it is a scalar quantity. The SI unit is the pascal (Pa), which is a pressure of one newton per square metre (N m^{-2}). Blaise Pascal (1623-1662) was a French mathematician and philosopher who contributed greatly to the theory of conic sections and to hydrostatics.

He showed that the barometric pressure decreases with height - hence the famous examination question: "Explain how you would use a barometer to measure the height of a tall building" - to which the most accurate answer is said to be: "I would drop it out of the window and time how long it took to reach the ground." The CGS unit of pressure is dyne cm^{-2} and $1 \text{ Pa} = 10 \text{ dyne cm}^{-2}$. Some other silly units for pressure are often seen, such as psi, bar, Torr or mm Hg, and atm. A psi or "pound per square inch" is all right for those who define a "pound" as a unit of force (US usage) but is less so for those who define a pound as a unit of mass (UK usage). A psi is about 6894.76 Pa. [The "British Engineering System", as far as I know, is used exclusively in the U.S. and is not and never has been used in Britain, where it would probably be unrecognized. In the "British" Engineering System, the pound is defined as a unit of force, whereas in Britain a pound is a unit of mass.] A bar is 105 Pa or 100 kPa. A Torr is a pressure under a column of mercury 760 mm high. This may be convenient for casual conversational use where extreme precision is not expected in laboratory experiments in which pressure is actually indicated by a mercury barometer or manometer. To find out exactly what the pressure in Pa under 760 mm Hg is, one would have to know the exact value of the local gravitational acceleration and also the exact density of mercury, which varies with temperature and with isotopic constitution. A Torr is usually given as 133.322 Pa. Evangelista Torricelli (1608 - 1647) is regarded as the inventor of the mercury barometer. He succeeded Galileo as professor of mathematics at the University of Florence. An atm is 760 torr or about 14.7 psi or 101 325 Pa. That is to say, 1.013 25 bar. As usual, the use of a variety of different units, and knowing the exact definitions and conversion factors between all of them and carrying out all the tedious multiplications, is an unnecessary chore that is inflicted upon all of us in all branches of physics.

4. Pressure on a Horizontal Surface. Pressure at Depth z FIGURE XVI. 1 z .4

Figure XVI. 1 shows a horizontal surface of area $.4$ immersed at a depth z under the surface of a fluid at depth z . The force F on the area $.4$ is equal to

the weight of the superincumbent fluid. This gives us occasion to use a gloriously pompous word. "Incumbent" means "lying down", so that "superincumbent" is lying down above the area. It is incumbent upon all of us to understand this. The weight of the superincumbent fluid is evidently its volume ρz times its density ρ times the gravitational acceleration g . Thus $F = \rho g z A$, 16.4.1 and, since pressure is force per unit area, we find that the pressure at a depth z is $P = \rho g z$. 16.4.2 This is, of course, in addition to the atmospheric pressure that may exist above the surface of the liquid. The pressure is the same at all points at the same horizontal level within a homogeneous incompressible fluid. This seemingly trivial statement may sometimes be worth remembering under the stress of examination conditions. Thus, let's look at an example. 0.8 ' D 155 cm 13.6 1.01 kPa A 200 cm B 160cm FIGURE XVI.2

In figure XVI.2, the vessel at the left is partly filled with a liquid of density 0.8 g cm^{-3} , the upper part of the vessel being filled with air. The liquid also fills the tube along to the point C. From C to B, the tube is filled with mercury of density 13.6 g cm^{-3} . Above that, from B to A, is water of density 1.0 g cm^{-3} and above that is the atmospheric pressure of 101 kPa . The height of the four interfaces above the thick black line are A 200 cm B 160 cm C 140 cm D 155 cm With $g = 9.8 \text{ m s}^{-2}$, what is the pressure of the air in the closed vessel? I'll do the calculation in SI units. Pressure at A = 101000 Pa Pressure at B = $101000 + 1000 \times 9.8 \times 0.4 = 104920 \text{ Pa}$ Pressure at C = $104920 + 13600 \times 9.8 \times 0.20 = 131576 \text{ Pa}$ Pressure at D = $131576 - 800 \times 9.8 \times 0.15 = 130400 \text{ Pa}$, and this is the pressure of the air in the vessel. Rather boring so far, and the next problem will also be boring, but the problem after that should keep you occupied arguing about it over lunch. Problem. This problem is purely geometrical and nothing to do with hydrostatics - but the result will help you with the next problem after this. If you don't want to do it, just use the result in the next problem. 1.3 Show that the volume of the frustrum of a cone, whose upper and lower circular faces are 1 2 of radii r and r_2 , and whose height is h , is $\frac{1}{3}h(r^2 + r r_2 + r_2^2)$

Problem. (Pascal's Paradox) $\tan \theta = 8/(15x/-)$ I 9 FIGURE XVI.4 Figure XVI.4 shows four vessels. $10 / \text{cm}$, so the area is 100 cm^2 . depth of 15 cm . The base of each is circular with the same radius, Each is filled with water (density = 1 g cm^{-3}) to a Calculate 1. The mass of water in each. 2. The

pressure at the bottom of each vessel. 3. The force on the bottom of each vessel.

If the bottom of each vessel were made of glass, which cracked under a certain pressure, which would crack first if the vessels were slowly filled up? If the bottom of each vessel were welded to the scale of a weighing machine, what weight would be recorded? I'll leave you to argue about this as long as you wish. 5. Pressure on a Vertical Surface FIGURE XVI.5 Figure XVI.5 shows a vertical surface from the side and face-on. The pressure increases at greater depths. I show a strip of the surface at depth z . Suppose the area of that strip is dA . The pressure at depth z is $p_g z$, so the force on the strip is $p_g z dA$. The force on the entire area is $p_g \int z dA$, and that, by definition of the centroid (see chapter 1), is $p_g \bar{z} A$, where $\bar{z}$ is the depth of the centroid. The same result will be obtained for an inclined surface. Therefore: The total force on a submerged vertical or inclined plane surface is equal to the area of the surface times the depth of the centroid. Example. Figure XVI.6 shows a triangular area. The uppermost side of the triangle is parallel to the surface at a depth z . The depth of the centroid is $z + h$, so the pressure at the centroid is $p_g(z + h)$. The area of the triangle is $\frac{1}{2}bh$, so the total force on the triangle is $p_g b h (z + h)$.

FIGURE XVI.6 6. Centre of Pressure "The centre of pressure is the point at which the pressure may be considered to act." This is a fairly meaningless sentence, yet it is not entirely devoid of all meaning. If you refer to the left hand side of figure XVI.5, you will see an infinite number (I've drawn only eight) of forces. If you were to replace all of these forces by a single force, where would you put it? Or, more precisely, if you were to replace all of these forces by a single force such that the (first) moment of this force about a line through the surface of the fluid is the same as the (first) moment of all the actual forces, where would you place this single force? You would place it at the centre of pressure. The depth of the centre of pressure is a depth such that the moment of the total force on a vertical surface about a line in the surface of the fluid is the same as the moment of all the hydrostatic forces about a line in the surface of the fluid. I shall use the Greek letter $\bar{z}$ to indicate the depth of the centre of pressure. We can continue to use figure XVI.5. The force on the strip of area dA at depth z is, as we have seen, $p_g z dA$, so the first moment of that force is $p_g z^2 dA$. The total moment is therefore $p_g \int z^2 dA$,

which is, by definition of radius of gyration k , (see chapter 2), pgk^2/A . The total force, as we have seen, is pgA , and the total moment is to be this times k^2 . Thus the depth of the centre of pressure is k^2 .

Example. FIGURE XVI.7 A semicircular trough of radius a is filled with water, density p . One semicircular end of the trough is freely hinged at its diameter (the thick line in the figure). What force must be exerted at the bottom of the trough to prevent the end from swinging open? 2 so the total The area of the semicircle is $\frac{1}{2}\pi a^2$. The depth of the centroid is $\frac{4a}{3}$. The hydrostatic force is $\frac{1}{2}\pi pga$. The square of the radius of gyration is $\frac{7a^2}{8}$, so the depth of the centre of pressure is $\frac{16a}{3}$. The moment of the hydrostatic forces is therefore $\frac{16}{3}\pi pga^2$. If the required force is F , this must equal Fa , and therefore $F = \frac{16}{3}\pi pga$.

16.7 Archimedes' Principle The most important thing about Archimedes' principle is to get the apostrophe in the right place and to spell principle correctly. Archimedes was a Greek scientist who lived in Syracuse, Sicily. He was born about 287 BC and died about 212 BC. He made many contributions to mechanics. He invented the Archimedean screw, he is reputed to have said "Give me a fulcrum and I shall move the world", and he probably did not set the Roman invading fleet on fire by focussing sunlight on them with concave mirrors - though it makes a good story. The most famous story about him is that he was commissioned by King Hiero of Sicily to determine whether the king's crown was contaminated with base metal. Archimedes realized that he would need to know the density of the crown. Measuring its weight was no problem, but - how to measure the volume of such an irregularly-shaped object? One day, he went to take a bath, and he had filled the bath full right to the rim. When he stepped into the bath he was much surprised that some of the water slopped over the edge of the bath on to the floor. Suddenly, he realized that he had the solution to his problem, so

straightway he raced out of the house and ran absolutely starkers through the streets of Syracuse shouting "EuprltOt! EuprltOt!", which is Greek for: When a body is totally or partially immersed in a fluid, it experiences a hydrostatic upthrust equal to the weight of fluid displaced. Figure XVI.8 is a drawing of some water or other fluid. I have outlined with a dashed curve an arbitrary portion of the fluid. It is subject to hydrostatic pressure from the rest of the fluid. The small pressure of the fluid above it is pushing it down; the larger

pressure of the fluid below it is pushing it up. Therefore there is a net upthrust. The portion of the fluid outlined is in equilibrium between its own weight and the hydrostatic upthrust. If we were to replace this portion of the fluid with a lump of iron, we wouldn't have changed the hydrostatic forces. Therefore the upthrust is equal to the weight of fluid displaced. FIGURE XVI.8 If a body is floating on the surface, the hydrostatic upthrust, as well as being equal to the weight of fluid displaced, is also equal to the weight of the body. 16.8 Some Simple Examples As we pointed out in the introduction to this chapter, this chapter is less demanding than some of the others, and indeed it has been quite trivial so far. Just to show how easy the topic is, here are a few quick examples. 1. A cylindrical vessel of cross-sectional area A is partially filled with water. A mass m of ice floats on the surface. The density of water is ρ_0 and the density of ice is ρ . Calculate the change in the level of the water when the ice melts, and state whether the water level rises or falls.

2. A cork of mass m , density ρ , is held under water (density ρ_0) by a string. Calculate the tension in the string. Calculate the initial acceleration if the string is cut. 3. A lump of lead (mass m , density ρ) is held hanging in water (density ρ_0) by two strings as shown. Calculate the tension in the strings. 4. A hydrometer (for our purposes a hydrometer is a wooden rod weighted at the bottom for stability when it floats vertically) floats in equilibrium to a depth z in water of density ρ . If salt is added to the water so that the new density is ρ_2 , what is the new depth z_2 ? 5. A mass m , density ρ , hangs in a fluid of density ρ_0 from the ceiling of an elevator (lift). The elevator accelerates upwards at a rate a . Calculate the tension in the string.

6. A hydrometer of mass m and cross-sectional area A floats in equilibrium to a depth h in a liquid of density ρ . The hydrometer is then gently pushed down and released. Determine the period of oscillation. 7. A rod of length l and density ρ ($\rho < \rho_0$) floats in a liquid of density ρ_0 . One end of the rod is lifted up through a height y so that a length x remains immersed. I have drawn it with the rope vertical. Must it be? i. Find x as a function of y . ii. Find 0 as a function of y and s . iii. Find the tension T in the rope as a function of m , g and s . Draw the following graphs: a. x and $T/(mg)$ versus s . b. 0 versus y for several s . c. 0 versus s for several y . d. x versus y for several s . e. $T/(mg)$ versus y for several s .

alliswet s 1. No, it doesn't. $P - P_o = mg \sin \theta \cos \theta + \dots \tan \theta$ 2. $P - P_o = mg \sin \theta$: $\cos \theta$ 2 + $\dots \tan \theta$ 4. $Z_2 : f)l Z_1$ 6. $P = 2irIg P$ 7. i. $x = 1 - x/-x$ ii. $\sin \theta = Y x/1 - s$ iii.

0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2 0.1 0 0 $/(mg)$ 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
s 0. 90 80 70 60 50 40 30 20 10 0 0 0.6 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 Y
0.9 1

Co 90 80 70 2 60 50 40 30 10 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 0.8 0.7
0.6 0.8 "" 0.5 0.4 0.6 0.4 0.2 0.2 0.1 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 Y 0.8 0.9

1 0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2 0.1 0 0 $T/(mg)$ s=0.2 0.4 0.6 0.8 0.1 0.2 0.3
0.4 0.5 0.6 0.7 0.8 0.9 1 Y 16.9 Floating Bodies This is the most grisly topic
in hydrostatics. We can start with an observation that we have already made
is section 16,7, namely that, if a body is freely floating, the hydrostatic
upthrust is equal to the weight of the body. I also introduce here the term
centre of buoyancy, which is the centre of mass of the displaced fluid. In a
freely-floating body in equilibrium, the centre of buoyancy is vertically
below the centre of mass of the floating body. As far as calculating the
moment about some axis of the hydrostatic upthrust is concerned, the
upthrust can be considered to act through the centre of buoyancy, just as the
weight of an object can be considered to act through its centre of mass. See
section 1.1 of chapter 1, for example, for a discussion of this point. Also,
before we get going, here is another small problem.

8. The drawing shows a body, whose relative density (i.e. relative to the fluid that it is floating in) is s . The dashed line is the water-line section. Now, in the next drawing, a body of exactly the same size and shape (but not necessarily the same density) is floating upside down, with the same water-line section. What is the relative density of this second body? I want to look now at the stability of equilibrium of a freely-floating body. While at first sight this may not be a very interesting topic, if you ever happen to be a passenger on an ocean liner, you might then find it to be quite interesting, for you will be interested to know, if the liner is given a small angular displacement from the vertical position, whether it will capsize and throw you into the sea, or whether it will right itself. Under such circumstances it becomes a very interesting subject indeed. Before I start, I just want to establish one small geometric result. FIGURE XVI.9

Figure XVI.9 shows a plane bilaterally-symmetric area. I have drawn a dashed line through the centroid of the area. The areas to the left and right of this line are A_1 and A_2 , and I have indicated the positions of the centroids of these two areas. (I haven't calculated the positions of the three centroids accurately - I just drew them approximately where I thought they would be.) Note that, since the dashed line goes through the centroid of the whole area, $A_1 = A_2$. Now rotate the area about the dashed line through an angle θ . By the theorem of Pappus (see chapter 1, section 1.6), the volume swept out by A_1 is $A_1 \times \theta$ and the volume swept out by A_2 is $A_2 \times \theta$. Thus we have established the geometrical result that I wanted, namely, that when a bilaterally symmetric area is rotated about an axis perpendicular to its axis of symmetry and passing through its centroid, the areas to left and right of the axis of rotation sweep out equal volumes. We can now return to floating bodies, and I am going to consider the stability of equilibrium of a bilaterally symmetric floating body to a rotational displacement about an axis lying in the water line section and perpendicular to the axis of symmetry. FIGURE XVI. 10 I have drawn in figure XVI. 10 the centre of mass C of the whole body, the centre of buoyancy H , and the centroid of the water-line section. The body is bilaterally symmetric about the plane of the paper, and we are going to rotate the body about an axis through O perpendicular to the plane of the paper, and we want to know whether the equilibrium is stable against such an angular displacement. We are going to rotate it in such a manner that the volume submerged is unaltered by the rotation - which means that the hydrostatic upthrust will remain equal to the weight of the body, and there will be no vertical acceleration. The geometrical theorem that we have just established shows that, if we

rotate the body about an axis through the centroid of the water-line section, the volume submerged will be constant; conversely, our condition that the volume submerged is constant implies that the rotation is about an axis through the centroid of the water-line section. I am going to establish a set of rectangular axes, origin O , with the x -axis to the right, the y -axis towards you, and the z -axis downwards. I'm going to call the depth of the centre H of buoyancy h . Now let's carry out the rotation about O through an angle θ . FIGURE XVI. 11 I have drawn the position of the new centre of buoyancy H' and I wish to find its coordinates (x', y') relative to O . We shall find that it has moved a little horizontally compared with the original position of H , but its

depth is almost unchanged. Indeed, for small θ , we shall find that h is of order θ^2 , while h' is of order θ . Thus, to first order in θ , I shall assume that the depth of the centre of buoyancy has remained unchanged. However, the coordinate h' of the new centre of buoyancy will be of interest for the following reason. The weight of the body acts at its centre of mass C while the hydrostatic upthrust acts at the new centre of buoyancy H' and these two forces form a couple and exert a torque. You will understand from figure XVI. 11 that if H' is to the left of C , the torque will topple the body over, whereas if H' is to the right of C , the torque will stabilize the body. Indeed, the horizontal distance between C and H' is known as the righting/ever. The point on the line COH vertically above H' is called the metacentre. I

haven't drawn it on the diagram, in order to minimise clutter, but I shall use the symbol M to indicate the metacentre. We can see that the condition for stability of equilibrium is that $HM > HC$. This is why we are interested in finding the exact position of the new centre of buoyancy H' . A x-axis y-axis x-axis FIGURE XVI. 12

In the upper part of figure XVI. 12 I have drawn the old and new water-line sections as seen from the side, and in the lower part I have drawn the new water-line section seen from above. I have indicated an elemental volume of width b_x of the displaced fluid at a distance x from the centroid O of the water-line section. For small θ the depth of this element is xO . Let's call its area in the water-line section δA , so that the volume element is $xO\delta A$. We'll call the total volume of the displaced fluid (which is unaltered by the rotation) V . Consider the moments of volume about the x-axis. We have $A' B' V' = V - \int_0^{\theta} xO \cdot xO \cdot \delta A + \int_0^{\theta} xO \cdot xO \cdot \delta A \cdot \sin \theta$. $r' = \frac{1}{V} \int_0^{\theta} xO \cdot xO \cdot \delta A \cdot \sin \theta$. Thus, as previously asserted, the vertical displacement of the centre of buoyancy is of order θ^2 , and, to first order in θ may be neglected. Now consider the moments of volume about the y-axis. We have $r'' = \frac{1}{V} \int_0^{\theta} xO \cdot \delta A \cdot x + \int_0^{\theta} xO \cdot \delta A \cdot x \cdot \sin \theta$. $V(r'' - r) = \int_0^{\theta} xO \cdot \delta A \cdot x \cdot \sin \theta$. But the integral on the right hand side of equation 16.9.2 is $Ak^2 \theta^2$, where A is the area of the water-line section, and k is its radius of gyration. $Ak^2 \theta^2 = V HM \sin \theta$. Thus $HM \sin \theta = \frac{Ak^2 \theta^2}{V}$. 16.9.3 $V HM \sin \theta$, where M is the metacentre, or, to first order in θ , $HM \sin \theta = HM \theta$. $Ak^2 \theta^2 = V HM \theta$. 16.9.4 V Therefore the condition for stability of equilibrium is that $Ak^2 > V HC$. 16.9.5 Here, A and k^2 refer to the water-line section, V is the volume submerged, and HC is the distance between centre of mass and centre

of buoyancy.

Example. Suppose that the body is a cube of side $2a$ and of relative density s . The water-line section is a square, and $I = 4a^2$ and $k = a^2/3$. The volume submerged is $8a^3s$. The distance between the centres of mass and buoyancy is $a(1 - s)$, and so the condition for stability is $-- > a(1 - s)$. 16.9.6 $6s$ The equilibrium is unstable if $6s^2 - 6s + 1 < 0$. 16.9.7 That is, the equilibrium is unstable if s is between 0.2113 and 0.7887. The cube will float vertically only if the density is less than 0.2113 or if it is greater than 0.7887. Here in British Columbia there is a large logging industry, and many logs float horizontally in the water. They gradually become waterlogged, and, when the density of a log is nearly as dense as the water, the vertical position becomes stable and the log tips to the vertical position, nearly all of it submerged, with only an inch or so above the surface. It then becomes a danger to boats. If the length of the log is $2l$ and its radius is a , what is the least relative density for which the vertical position is stable?

CHAPTER 17 VIBRATING SYSTEMS 17.1 Introduction A mass m is attached to an elastic spring of force constant k , the other end of which is attached to a fixed point. The spring is supposed to obey Hooke's law, namely that, when it is extended (or compressed) by a distance x from its natural length, the tension (or thrust) in the spring is kx , and the equation of motion is $m\ddot{x} = -kx$. This is simple harmonic motion of period $2\pi/\omega$, where $\omega^2 = k/m$. Most readers will have no difficulty with that problem. But now suppose that, instead of one end of the spring being attached to a fixed point, we have two masses, m and m_2 , one at either end of the spring. A diatomic molecule is much the same thing. Can you calculate the period of simple harmonic oscillations? It looks like an easy problem, but it somehow seems difficult to get a hand on it by conventional newtonian methods. In fact it can be done quite readily by newtonian methods but this problem, as well as more complicated problems where you have several masses connected by several springs and several possible modes of vibration, is particularly suitable by lagrangian methods, and this chapter will give several examples of vibrating systems tackled by lagrangian methods. 17.2 The Diatomic Molecule Two particles, of masses m_1 and m_2 are connected by an elastic spring of force constant k . What is the period of oscillation? FIGURE XVII. 1 m_1 k m_2 x_1 x_2 Let's suppose that the equilibrium separation of the masses - i.e. the natural,

unstretched, uncompressed length of the spring - is a . At some time suppose that the x -coordinates of the two masses are x and x_2 . The extension q of the spring from its natural length at that

moment is $q = x_2 - x - a$. We'll also suppose that the velocities of the two masses at that instant are $\dot{x}$ and $\dot{x}_2$. We know from chapter 13 how to start any calculation in lagrangian mechanics. We don't have to think about it. We always start with $T = \dots$ and $V = \dots$. $T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{x}_2^2$, 17.2.1 17.2.2 We want to be able to express the equations in terms of the internal coordinate q , already expressed in terms of q . Now we need to express T (and therefore and in terms of q . Since $q = x_2 - x - a$, we have, by differentiation with respect to time, $\dot{q} = \dot{x}_2 - \dot{x}$. We need one more equation. The linear momentum is constant and there is no loss in generality in choosing a coordinate system such that the linear momentum is zero: $0 = m\dot{x} + m\dot{x}_2$. 17.2.4 From these two equations, we find that $\dot{x}_2 = -\dot{x}$ and $\dot{q} = \dot{x}$. 17.2.5a,b $m\dot{x} + m\dot{x}_2 = 0$ 17.2.5c $\dot{x}_2 = -\dot{x}$ 17.2.6 Thus we obtain $T = \frac{1}{2} m \dot{q}^2$ 17.2.6 and $V = \frac{1}{2} k q^2$, 17.2.2 $\frac{1}{2} m \dot{q}^2$ where $m = \frac{m_1 m_2}{m_1 + m_2}$ 17.2.7 Now apply Lagrange's equation $\frac{d}{dt} \frac{\partial V}{\partial \dot{q}} = -\frac{\partial V}{\partial q}$ 13.4.13 to the single coordinate q in the fashion to which we became accustomed in Chapter 13, and the equation of motion becomes

$m \ddot{q} = -kq$, 17.2.8 which is simple harmonic motion of period $2\pi/\omega$, where m is given by equation 17.2.7. The frequency is the reciprocal of this, and the "angular frequency" ω , also sometimes called the "pulsatance", is 2 times the frequency, or $\omega = 2\pi/\text{period}$. The quantity $m = m_1 m_2 / (m_1 + m_2)$ is usually called the "reduced mass" and one may wonder in what sense it is "reduced". I believe the origin of this term may come from an elementary treatment of the Bohr atom of hydrogen, in which one at first assumes that there is an electron moving around an immovable nucleus - i.e. a nucleus of "infinite mass". One develops formulas for various properties of the atom, such as, for example, the Rydberg constant, which is the energy required to ionize the atom from its ground state. This and similar formulas include the mass m of the electron. Later, in a more sophisticated model, one takes account of the finite mass of the nucleus, with nucleus and electron moving around their mutual centre of mass. One arrives at the same formula, except that m is replaced by $mM/(m + M)$, where M is the mass of the nucleus. This is slightly less (by about 0.05%) than the mass of the electron, and the idea is that you can do the calculation with a fixed nucleus provided that you use this "reduced mass" of

the electron" rather than its true mass. Whether this is the appropriate term to use in our present context is debatable, but in practice it is the term almost universally used. It may also be remarked upon by readers with some familiarity with quantum mechanics that I have named this section "The Diatomic Molecule" - yet I have ignored the quantum mechanical aspects of molecular vibration. This is true - in this series of notes on Classical Mechanics I have adopted an entirely classical treatment. It would be wrong, however, to assume that classical mechanics does not apply to a molecule, or that quantum mechanics would not apply to a system consisting of a cricket ball and a baseball connected by a metal spring. In fact both classical mechanics and quantum mechanics apply to both. The formula derived for the frequency of vibration in terms of the reduced mass and the force constant ("bond strength") applies as accurately for the molecule as for the cricket ball and baseball. Quantum mechanics, however, predicts that the total energy (the eigenvalue of the hamiltonian operator) can take only certain discrete values, and also that the lowest possible value is not zero. It predicts this not only for the molecule, but also for the cricket ball and baseball - although in the latter case the energy levels are so closely spaced together as to form a quasi continuum, and the zero point vibrational energy is so close to zero as to be unmeasurable. Quantum mechanics makes its effects evident at the molecular level, but this does not mean that it does not apply at macroscopic levels. One might also take note that one is not likely to understand why wave mechanics predicts only discrete energy levels unless one has had a good background in the classical mechanics of waves. In other words, one must not assume that classical mechanics does not apply to microscopic systems, or that quantum mechanics does not apply to macroscopic systems. Below leaving this section, in case you tried solving this problem by newtonian methods and ran into difficulties, here's a hint. Keep the centre of mass fixed. When the length of the spring is x , the lengths of the portions on either side of the centre of mass are m_1x and m_2x . The force constants of the two portions of the spring are k_1 and k_2 : inversely proportional to their lengths. Take it from there.

17.3 Two Masses, Two Springs and a Brick Wall

The system is illustrated in figure XVII.2, first in its equilibrium (unstretched) position, and then at some instant when it is not in equilibrium and the springs are stretched. You

4 can imagine that the masses are resting upon and can slide upon a smooth,

horizontal table. I could also have them hanging under gravity, but this would introduce a distracting complication without illustrating any further principles. I also want to assume that all the motion is linear, so we could have them sliding on a smooth horizontal rail, or have them confined in the inside of a smooth, fixed drinking-straw. For the present, I don't want the system to bend. I k m k₂ m₂ Equilibrium: $x_1 = 0, x_2 = 0$, Displaced: $x_1 = x, x_2 = x_2$ FIGURE XVII.2 The displacements from the equilibrium positions are x and x_2 , so that the two springs are stretched by x and $x_2 - x$ respectively. The velocities of the two masses are $\dot{x}$ and $\dot{x}_2$. We now start the lagrangian calculation in the usual manner: 17.3.1 $T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m_2 \dot{x}_2^2$ 17.3.2 $V = \frac{1}{2} k x^2 + \frac{1}{2} k_2 (x_2 - x)^2$. Apply Lagrange's equation to each coordinate in turn, to obtain the following equations of motion: 17.3.3 $m \ddot{x} = -(k + k_2)x + k_2 x_2$ and $m_2 \ddot{x}_2 = k_2 x - k_2 x_2$. 17.3.4 Now we seek solutions in which the system is vibrating in simple harmonic motion at angular frequency ω ; that is, we seek solutions of the form $x = A \cos \omega t$ and $x_2 = B \cos \omega t$. When we substitute these in equations 17.3.3 and 4, we obtain $(k + k_2 - m\omega^2)x - k_2 x_2 = 0$ 17.3.5 and $k_2 x - (k_2 - m_2\omega^2)x_2 = 0$. 17.3.6

Either of these gives us the displacement ratio x_2/x (and hence amplitude ratio). The first gives us $x_2/x = -m_2\omega^2 / (k + k_2 - m\omega^2)$ 17.3.7 and the second gives us $x_2/x = k_2 / (k_2 - m_2\omega^2)$ 17.3.8 These are equal, and, by equating the right hand sides, we obtain the following equation for the angular frequencies of the normal modes: $m_2^2\omega^4 - (mk_2 + m_2k + k_2^2)\omega^2 + kk_2 = 0$. 17.3.9 This equation can also be derived by noting, from the theory of equations, that equations 17.3.5 and 6 are consistent only if the determinant of the coefficients is zero. The meaning of these equations and of the expression "normal modes" can perhaps be best illustrated with a numerical example. Let us suppose, for example, that $k = k_2 = 1$ and $m = 3$ and $m_2 = 2$. In that case equation 17.3.9 is $6\omega^4 - 7\omega^2 + 1 = 0$. This is a quartic equation in ω , but it is also a quadratic equation in ω^2 , and there are just two positive solutions for ω . These are $1/\omega = 0.4082$ (slow, low frequency) and 1 (fast, high frequency). If you put the high frequency ω into either of equations 17.3.7 or 8 (or in both, to check for arithmetic or algebraic mistakes) you find a displacement ratio of +1.5; but if you put the low frequency ω into either equation, you find a displacement ratio of -1.0 The first of these normal modes is a low-frequency slow oscillation in which the two masses oscillate in phase, with m_2 having an amplitude 50% larger than m . The second

normal mode is a high-frequency fast oscillation in which the two masses oscillate out of phase but with equal amplitudes. So, how does the system actually oscillate? This depends on the initial conditions. For example, if you stretch the first spring by one inch and the second spring by 1.5 inches, and then let go, the system will oscillate in the slow, in-phase mode. But if you start by stretching the first spring by one inch and compressing the second by one inch, the system will oscillate in the fast, out-of-phase mode. For other initial conditions, the system will oscillate in a linear combination of the normal modes. Thus, m_1 might oscillate with an amplitude A in the slow mode, and an amplitude B in the fast mode: $x_1 = A \cos(\omega_1 t + \phi_1) + B \cos(\omega_2 t + \phi_2)$, 17.3.10 in which case the oscillation of m_2 is given by $x_2 = 1.5A \cos(\omega_1 t + \phi_1) - B \cos(\omega_2 t + \phi_2)$. 17.3.11

In our example, ω_1 and ω_2 are $1/\sqrt{m}$ and 1 respectively. Let's suppose that the initial conditions are that, at $t = 0$, x_1 and x_2 are both zero. This means that ϕ_1 and ϕ_2 are both zero or π (I'll take them to be zero), so that $x_1 = A \cos \omega_1 t + B \cos \omega_2 t$. 17.3.12 $x_2 = 1.5A \cos \omega_1 t - B \cos \omega_2 t$. 17.3.13 Suppose further that at $t = 0$, $\dot{x}_1$ and $\dot{x}_2$ are both $+1$, which means that we start by stretching both springs equally. Equations 17.3.12 and 13 then become $1 = A + B$ and $1 = 1.5A - B$. That is, $A = 0.8$ and $B = 0.2$. I'll leave you to draw graphs of x_1 and x_2 versus time. 17.4 Double Torsion Pendulum FIGURE XVII.3 I1 C2 Here we have two cylinders of rotational inertias I_1 and I_2 hanging from two wires of torsion constants c_1 and c_2 . At any instant, the top cylinder is turned through an angle θ_1 from its equilibrium position and the lower cylinder by an (additional) angle θ_2 . The equations and the description of the motion are just the same as in the previous example, except that x , x_2 , m , m_2 , k , k_2 are replaced by θ , θ_2 , I_1 , I_2 , c_1 , c_2 . The kinetic and potential energies are $T = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2$, 17.4.1

$= \frac{1}{2} c_1 \theta_1^2 + \frac{1}{2} c_2 (\theta_2 - \theta_1)^2$. 17.4.2 The equations for θ_1 and the displacement ratios are just the same, and there is an in-phase and an out-of-phase mode. 17.5 Double Pendulum This is another similar problem, though, instead of assuming Hooke's law, we shall assume that angles are small ($\sin \theta \approx \theta$, $\cos \theta \approx 1 - \frac{1}{2} \theta^2$). For clarity of drawing, however, I have drawn large angles in figure XVIII.4. 1 FIGURE XVII.4 θ_2 [2(2 Because I am going to use the lagrangian equations of motion, I have not marked in the forces and accelerations; rather, I have marked in the velocities. I hope that the two

components of the velocity of m_2 that I have marked are self-explanatory; the speed of m_2 $v^2 = \dot{x}_2^2 + \dot{y}_2^2$ is given by $v^2 = \dot{x}_1^2 + \dot{y}_1^2 + 2\dot{x}_1\dot{x}_2\cos(\theta_2 - \theta_1) + \dot{y}_1^2 + \dot{y}_2^2 + 2\dot{y}_1\dot{y}_2\cos(\theta_2 - \theta_1)$. The kinetic and potential energies are $T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_2\dot{y}_2^2 + m_1gl(1 - \cos\theta_1) + m_2gl(1 - \cos\theta_2)$. 17.5.1 $T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2[\dot{x}_1^2 + \dot{y}_1^2 + 2\dot{x}_1\dot{x}_2\cos\theta_1 + \dot{y}_1^2 + \dot{y}_2^2 + 2\dot{y}_1\dot{y}_2\cos(\theta_2 - \theta_1)]$. 17.5.2 If we now make the small angle approximation, these become

17.5.3 $T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2[\dot{x}_1^2 + \dot{y}_1^2 + 2\dot{x}_1\dot{x}_2\cos\theta_1 + \dot{y}_1^2 + \dot{y}_2^2 + 2\dot{y}_1\dot{y}_2\cos(\theta_2 - \theta_1)]$ and $V = m_1gl(1 - \cos\theta_1) + m_2gl(1 - \cos\theta_2)$. Apply the Lagrangian equation in turn to θ_1 and θ_2 : and $(m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2l_2l_1\ddot{\theta}_2\cos\theta_1 = -(m_1 + m_2)gl\sin\theta_1$. 17.5.5 $m_2l_2\ddot{\theta}_2 + m_1l_2\ddot{\theta}_1 = -m_2gl\sin\theta_2$. 17.5.6 Seek solutions of the form $\theta_1 = \alpha\cos\omega t$ and $\theta_2 = \beta\cos\omega t$. Then $(m_1 + m_2)(l_1\alpha - g)\cos\omega t + m_2l_2\beta\cos\omega t = 0$. 17.5.7 and $l_2\beta\cos\omega t + (l_1\alpha - g)\cos\omega t = 0$. 17.5.8 Either of these gives the displacement ratio θ_2/θ_1 . ratio θ_2/θ_1 , or putting the determinant of the coefficients to zero, gives the following equation for the frequencies of the normal modes: Equating the two expressions for the 17.5.9 $m_1l_2\alpha^4 - (m_1 + m_2)gl(l_1 + l_2)\alpha^2 + (m_1 + m_2)g^2 = 0$. As in the previous examples, there is a slow in-phase mode, and fast out-of-phase mode. For example, suppose $m_1 = 0.01$ kg, $m_2 = 0.02$ kg, $l_1 = 0.3$ m, $l_2 = 0.6$ m, $g = 9.8$ m/s². Then $0.0018\alpha^4 - 0.2646\alpha^2 + 2.8812 = 0$. The slow solution is $\alpha = 3.441$ rad/s ($P = 1.826$ s), and the fast solution is $\alpha = 11.626$ rad/s ($P = 0.540$ s). If we put the first of these (the slow solution) in either of equations 17.5.7 or 8 (or both, as a check against mistakes) we obtain the displacement ratio $\theta_2/\theta_1 = 1.319$, which is an in-phase mode. If we put the second (the fast solution) in either equation, we obtain $\theta_2/\theta_1 = -0.5689$, which is an out-of-phase mode. If you were to start with $\theta_2/\theta_1 = 1.319$ and let go, the pendulum would swing in the slow in-phase mode. If you were to start with $\theta_2/\theta_1 = -0.5689$ and let go, the pendulum would swing in the fast out-of-phase mode. Otherwise the motion would be a linear combination of the normal modes, with the fraction of each determined by the initial conditions, as in the example in section 17.3. 17.6 Linear Triatomic Molecule In Chapter 2, Section 2.9, we discussed a rigid triatomic molecule. Now we are going to discuss three masses held together by springs, of force constants k_1 and k_2 . We are going

to allow it to vibrate, but not to rotate. Also, for the time being, I don't want the molecule to bend, so we'll put it inside a drinking straw so that all the vibrations are linear. By the way, for real triatomic molecules, the force

constants and rotational inertias are such that molecules vibrate much faster than they rotate. To see their vibrations you look in the near infra-red spectrum; to see their rotation, you have to go to the far infrared or the microwave spectrum.

FIGURE XVII.5 Suppose that the equilibrium separations of the atoms are a and a_2 . Suppose that at some instant of time, the x -coordinates (distances from the left hand edge of the page) of the three atoms are x, x_2, x_3 . The extensions from the equilibrium distances are then $q = x - x - a$, $q_2 = x_2 - x_2 - a_2$. We are now ready to start: 17.6.1 $T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$, 17.6.2 We need to express the kinetic energy in terms of the internal coordinates, and, just as for the diatomic molecule (Section 17.2), the relevant equations are 17.6.3 $x = a + q$, $x_2 = a_2 + q_2$, 17.6.4 and 0 = $m + m_2 + m_3$. 17.6.5 These can conveniently be written 17.6.6 By one dexterous flick of the fingers (!) we invert the matrix to obtain

17.6.7 where $M = m + m_2 + m_3$. On putting these into equation 17.6.1, we now have 17.6.8 $T = \frac{1}{2} M \dot{q}^2 + \frac{1}{2} m_2 \dot{q}_2^2$ and 17.6.9 $V = \frac{1}{2} k q^2 + \frac{1}{2} k_2 q_2^2$, 17.6.10 where $a = m(m_2 + m_3)/M$, $h = m_3 m/M$, $b = m_3(m + m_2)/M$ and, for future reference, $ab - h^2 = mm_2 m_3 / M = m_2 h$. 17.6.11 17.6.12 On application of Lagrange's equation in turn to the two internal coordinates we obtain 17.6.13 $a\ddot{q} + h\ddot{q}_2 + kq = 0$ and 17.6.14 $b\ddot{q}_2 + h\ddot{q} + k_2 q_2 = 0$. 17.6.15 Seek solutions of the form $q = q_0 \cos \omega t$ and $q_2 = q_2 \cos \omega t$ and we obtain the following two expressions for the extension ratios: 17.6.16 $q/q_2 = \frac{h\omega^2 - k_2}{k - h\omega^2}$. Equating them gives the equation for the normal mode frequencies: $(ab - h^2)\omega^4 - (ak_2 + bk)\omega^2 + kk_2 = 0$. 17.6.17 For example, if $k = k_2 = k$ and $m = m_2 = m_3$, we obtain, for the slow symmetric ("breathing") mode, $q/q_2 = +1$ and $\omega^2 = k/m$. For the fast asymmetric mode, $q/q_2 = -1$ and $\omega^2 = 3k/m$.

11 Example. Consider the linear OCS molecule whose atoms have masses 16, 12 and 32. Suppose that the angular frequencies of the normal modes, as determined from infrared spectroscopy, are 0.900 934 905 and 0.413 148 033. (I just made these numbers up, in unstated units, just for the purpose of illustrating the calculation. Without searching the literature, I can't say what they are in the real OCS molecule.) Determine the force constants. The key to doing it is equation 17.6.16. You know the constants a, b and c (they are just

functions of the masses), so just put in the value of 0) and solve for k and k_2 . It is immediately obvious that you cannot do this - you have one equation in two unknowns. So - what to do? Recall what we did in Section 2.9. We made an isotopic substitution - aO for 6O. This presumably doesn't change either the bond lengths or their strengths, so we obtain a second equation. Here, then, is a table in which I have listed for ^{16}OCS and for ^{18}OCS , the several functions of the masses, and I have also added supposed frequencies for the heavier isotopomer (which are slower than for the lighter isotopomer). ^{16}OCS ^{18}OCS Fast 0) 0.900 934 905 0.896 513 236 Slow 0) 0.413 148 033 0.397 911 376 m m_2 m_3 16 12 32 18 12 32 M 60 62 a 11.73 12.774 193 55 h 8.53 9.290 322 581 b 14.93 15.483 870 97 $ab - h$ 2 102.4 111.483 871 0 From the theory of quadratic equations, the sum of the solutions for 0) 2 for the quadratic equation $17.6.16$ is $ak_2 + bk$ and the product is $kk_2 ab - h$, $ab - h$ $\diamond$ Thus we can calculate $ak_2 + bk$ and kk_2 for each isotopomer: $ak_2 + bk = 100.5952$ 107.255 225 8 $kk_2 = 14.187$ 248 03 14.187 248 01 For each isotopomer, then, we have two simultaneous equations for k and k_2 - but one of these equations is quadratic, so we get two possibilities for the force constants. Thus for ^{16}OCS the solutions are $k = 3.812286$ $k_2 = 3.721$ 454

12 or $k_1 = 2.924$ 000 $k_2 = 4.852$ 000 So, which one is right? For ^{18}OCS the solutions are $k_1 = 4.002$ 900 $k_2 = 3.544$ 242 or $k = 2.924$ 000 $k_2 = 4.852$ 000 Only the latter solution is common to both isotopomers, so this is the correct one. Note that in this section we considered a linear triatomic molecule that was not allowed either to rotate or to bend, whereas in Chapter 2 we considered a rigid triatomic molecule that was not allowed either to vibrate or to bend. If all of these restrictions are removed, the situation becomes rather more complicated. If a rotating molecule vibrates, the moving atoms, in a co-rotating reference frame, are subject to the Coriolis force, and hence they do not move in a straight line. Further, as it vibrates, the rotational inertia changes periodically, so the rotation is not uniform. If we allow the molecule to bend, the middle atom can oscillate up and down in the plane of the paper (so to speak) or back and forth at right angles to the plane of the paper. These two motions will not necessarily have either the same amplitude or the same phase. Consequently the middle atom will whirl around in a Lissajous ellipse, giving rise to what has been called "vibrational angular momentum". In a real triatomic molecule, the vibrations are usually much faster than the relatively slow, ponderous rotation, so that vibration-rotation interaction is small - but

is by no means negligible and is readily observed in the spectrum of the molecule. 17.7 Two Masses, Three Springs, Two brick Walls The three masses are equal, and the two outer springs are identical. Figure XVII.6 shows the equilibrium position. $m \quad k_2 \quad m \quad k_1 \quad \dots \rightarrow x \quad \dots \rightarrow y$ FIGURE XVII.6 Suppose that at some instant the first mass is displaced a distance x to the right and the second mass is displaced a distance y to the right. The extensions of the first two springs are x and $y - x$ respectively, and the compression of the third spring is y . If the speeds of the masses are $\dot{x}$ and $\dot{y}$, we have for the kinetic and potential energies:

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 \quad V = \frac{1}{2} k x^2 + \frac{1}{2} k_2 (y - x)^2 + \frac{1}{2} k y^2.$$

17.7.1 and 17.7.2 Apply Lagrange's equation in turn to x and to y . $m \ddot{x} + (k + k_2)x - k_2 y = 0$ 17.7.3 and $m \ddot{y} + (k + k_2)y - k_2 x = 0$. 17.7.4 Seek solutions of the form $x = A \cos(\omega t)$ and $y = B \cos(\omega t)$. $(-m\omega^2 + k + k_2)x - k_2 y = 0$ 17.7.5 and $-k_2 x + (-m\omega^2 + k + k_2)y = 0$. 17.7.6 On putting the determinant of the coefficients to zero, we find for the frequencies of the normal modes $\omega^2 = k$ and $\omega^2 = k + 2k_2$. 17.7.7a,b m corresponding to displacement ratios $x/y = 1$ and $x/y = -1$. 17.7.8a,b Y/Y In the first, slow, mode, the masses move in phase and there is no extension or compression of the connecting spring. In the second, fast, mode the masses move in antiphase and the compressions or extension of the coupling spring is twice the extensions or compressions of the outer springs. The general motion is a linear combination of the normal modes: $x = A \cos(\omega_1 t + \phi_1) + B \cos(\omega_2 t + \phi_2)$, 17.7.9 17.7.10 17.7.11 $y = A \cos(\omega_1 t + \phi_1) - B \cos(\omega_2 t + \phi_2)$, $\dot{x} = -A \omega_1 \sin(\omega_1 t + \phi_1) - B \omega_2 \sin(\omega_2 t + \phi_2)$, $\dot{y} = -A \omega_1 \sin(\omega_1 t + \phi_1) + B \omega_2 \sin(\omega_2 t + \phi_2)$. 17.7.12

14 Suppose that the initial condition is at $t = 0$, $y = \dot{y} = 0$, $x = x_0$, $\dot{x} = 0$. That is, we pull the first mass a little to the right (keeping the second mass fixed) and then we let go. The second two equations establish that $c_1 = c_2 = 0$, and the first two equations tell us that $t = B = x_0/2$. The displacements are then given by $x = \frac{1}{2} x_0 (\cos \omega_1 t + \cos \omega_2 t)$: $x_0 \cos \frac{1}{2}(\omega_1 + \omega_2)t \cos \frac{1}{2}(\omega_1 - \omega_2)t$ 17.7.13 and $y = \frac{1}{2} x_0 (\cos \omega_1 t - \cos \omega_2 t)$: $-x_0 \sin \frac{1}{2}(\omega_1 + \omega_2)t \sin \frac{1}{2}(\omega_1 - \omega_2)t$. 17.7.14 Let us imagine, for example, that k_2 is much less than k (but not negligible), so that we have two weakly-coupled oscillators. In that case equations 17.7.7 tell us that the frequencies of the two normal modes are nearly equal. What equation 17.7.13 describes, then, is a rapid oscillation of the first mass with angular frequency $\frac{1}{2}(\omega_1 + \omega_2)$ whose amplitude is modulated with a slow

angular frequency $\omega = \sqrt{g/a}$. Equation 17.7.14 describes the same sort of motion for the second mass, except that the modulation is out of phase by 90° with the modulation of the motion of the first mass. For a while the first mass will oscillate with a large amplitude. This will gradually decrease, while the amplitude of the motion of the second mass increases until the motion of the first mass momentarily ceases. After that, the amplitude of the motion of the second mass starts to decrease, while the first mass starts up again. And so the motion continues, with the first mass and the second mass alternately taking up the motion.

17.8 Transverse Oscillations of Masses on a Taut String

FIGURE XVII.7 A light string of length $4a$ is held taut, under tension F between two fixed points. Three equal masses m are attached at equidistant points along the string. They are set into transverse oscillation of small amplitudes, the transverse displacements of the three masses at some time being y , y_2 and y_3 . The kinetic energy is easy. It is just $\frac{1}{2} m (\dot{y}^2 + \dot{y}_2^2 + \dot{y}_3^2)$.

15 The downward component of the force on the first mass is $F \cos \theta$, where $\cos \theta = Y/a$, which, to first order of small quantities, is just y/a . The potential energy of this mass (i.e. the work required to bring it to this position from its equilibrium position) is therefore $Fy - \frac{1}{2} Fy^2/a$. Similarly, the potential energy of the third mass is $Fy_3 - \frac{1}{2} Fy_3^2/a$. The downward component of the two forces on the second mass, similarly, is $F(y_2 - y)/a + F(y_2 - y_3)/a$. Thus the potential energy of the system is $V = \frac{1}{2} F/a (y^2 + y_2^2 + y_3^2 - 2yy_2 - 2yy_3 + 2y_2y_3)$.

17.8.1 Apply Lagrange's equation in turn to the three coordinates: $\frac{1}{2} m \ddot{y} + F(2y - y_2) = 0$, $\frac{1}{2} m \ddot{y}_2 + F(-y + 2y_2 - y_3) = 0$, $\frac{1}{2} m \ddot{y}_3 + F(-y + 2y_3 - y_2) = 0$. 17.8.5 Seek solutions of the form $y = A \cos \omega t$, $y_2 = B \cos \omega t$, $y_3 = C \cos \omega t$. Then $(2F - am\omega^2)y - Fy_2 = 0$, $-Fy + (2F - am\omega^2)y_2 - Fy_3 = 0$, $-Fy + (2F - am\omega^2)y_3 - Fy_2 = 0$. 17.8.8 Putting the determinant of the coefficients to zero gives an equation for the frequencies of the normal modes. The solutions are: Slow Medium Fast $\omega = \sqrt{g/a}$, $\omega = \sqrt{3g/a}$, $\omega = \sqrt{5g/a}$. Substitution of these into equations 17.8.6 to 8 gives the following displacement ratios: $y : y_2 : y_3 = 1 : 1 : 1$, $1 : 0 : -1$, $1 : -2 : 1$.

16 These are illustrated in figure XVII.8. As usual, the general motion is a linear combination of the normal modes, the relative amplitudes and phases of the modes depending upon the initial conditions. If the motion of the first

mass is a combination of the three modes with relative amplitudes in the proportion $c_1 : c_2 : c_3$, and with initial phases ϕ_1, ϕ_2, ϕ_3 , its motion is described by $y = c_1 \sin(\omega t + \phi_1) + c_2 \sin(\omega t + \phi_2) + c_3 \sin(\omega t + \phi_3)$. 17.8.9 $\omega = 1.85 \text{ rad/s}$ $c_1 = 2.41 \text{ m}$
FIGURE XVII.8

17 The motions of the second and third masses are then described by $Y_2 = X/l \sin(\omega t + \phi_1) - \frac{1}{3} \sin(\omega t + \phi_2)$ and $Y_3 = C \sin(\omega t + \phi_1) - \frac{1}{2} \sin(\omega t + \phi_2) + \frac{1}{3} \sin(\omega t + \phi_3)$. These can be written $Y = q_1 + q_2 + q_3$, 17.8.10 17.8.11 17.8.12 17.8.13 and $Y = q_1 - q_2 + q_3$, 17.8.14 where the q_i , like the y_i , are time-dependent coordinates. We could, if we wish, express the q in terms of the y , by solving these equations: 17.8.15 $q_1 = \frac{1}{2}(Y_1 + Y_2)$, 17.8.16 $q_2 = \frac{1}{2}(Y_1 - Y_2)$, 17.8.17 and $q_3 = \frac{1}{3}(Y_1 + Y_2 + Y_3)$. We have hitherto described the state of the system as a function of time by giving the values of the coordinates y, y_2 and y_3 . We could equally well, if we wished, describe the state of the system by giving, instead, the values of the coordinates q, q_2 and q_3 . Indeed it turns out that it is very useful to do so, and these coordinates are called the normal coordinates, and we shall see that they have some special properties. Thus, if you express the kinetic and potential energies in terms of the normal coordinates, you get 17.8.18 $T = \frac{1}{2} m (\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2)$ and $V = \frac{1}{2} F [(2 - x/l)q_1^2 + q_2^2 + (2 + x/l)q_3^2]$. 17.8.19 Note that there are no cross terms. When you apply Lagrange's equation in turn to the three normal coordinates, you obtain 17.8.20

18 $m \ddot{q}_1 = -2Fq_1$ 17.8.21 and $m \ddot{q}_3 = -(2 + x/l)Fq_3$. 17.8.22 Notice that the normal coordinates have become completely separated into three independent equations and that each is of the form $\ddot{q} = -\omega^2 q$ and that each of the normal coordinates oscillates with one of the frequencies of the normal modes. Much of the art of solving problems involving vibrating systems concerns identifying the normal coordinates. 17.9 Vibrating String It is possible that the three modes of vibration of the three masses in section 17.8 reminded you of the fundamental and first two harmonic vibrations of a stretched string - and it is quite proper that it did. If you were to imagine ten masses attached to a stretched string and to carry out the same sort of analysis, you would find ten normal modes, of which one would be quite like the fundamental mode of a stretched string, and the remainder would remind you of the first nine harmonics. You could continue with the same analysis but with a very large number of masses, and eventually you would be analysing the vibrations of a

continuous heavy string. We do that now, and we assume that we have a heavy, taut string of mass μ per unit length, and under a tension F . Y F F A_0 B_0 FIGURE XVII.9 x I show in figure XVII.9 a portion of length l of a vibrating rope, represented by A_0B_0 in its equilibrium position and by AB in a displaced position. The rope makes an angle

θ_A with the horizontal at A and an angle θ_B with the horizontal at B . The tension in the rope is F . The vertical equation of motion is $F(\sin\theta_B - \sin\theta_A) = \mu \frac{\partial^2 y}{\partial t^2}$. 17.9.1 If the angles are small, then $\sin\theta \approx \theta$, so the expression in parenthesis is $-\frac{\partial^2 y}{\partial x^2}$. The equation of motion is therefore $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, where $c = \sqrt{F/\mu}$. 17.9.2,a,b As can be verified by substitution, the general solution to this is of the form $y = f(x - ct) + g(x + ct)$. 17.9.3 This represents a function that can travel in either direction along the rope at a speed c given by equation 17.9.2b. Should the disturbance be a periodic disturbance, then a wave will travel along the rope at that speed. Further analysis of waves in ropes and strings is generally done in chapters concerned with wave motion. This section, however, at least establishes the speed at which a disturbance (periodic or otherwise) travels along a stretched string or rope. 17.10 Water Water consists of a mass M ("oxygen") connected to two smaller equal masses m ("hydrogen") by two equal springs of force constants k , the angle between the springs being 2θ . The equilibrium length of each spring is r . The torque needed to increase the angle between the springs by $2\theta_0$ is $2c\theta_0$. See figure XVII. 10. FIGURE XVII. 10

20 At any time, let the coordinates of the three masses (from left to right) be (X_1, Y_1) , (X_2, Y_2) , (X_3, Y_3) and let the equilibrium positions be (X_{10}, Y_{10}) , (X_{20}, Y_{20}) , (X_{30}, Y_{30}) , where $Y_{30} = Y_{10}$. We suppose that these coordinates are referred to a frame in which the centre of mass of the system is stationary. q_1 q_2 $2mq_2^2/M$ $q_2 \sin\theta$ q_3 FIGURE XVII. 11

21 Let us try and imagine, in figure XVII. 11, the vibrational modes. We can easily imagine a mode in which the angle opens and closes symmetrically. Let us resolve this mode into an x -component and a y -component. In the x -component of this motion, one hydrogen atom moves to the right by a distance q while the other moves to the left by an equal distance q . In the y -component of this symmetric motion, both hydrogens move upwards by a distance $q/2$, while, in order to keep the centre of mass of the system

unmoved, the oxygen necessarily moves down by a distance $2mq^2/M$. We can also imagine an asymmetric mode in which one spring expands while the other contracts. One hydrogen moves down to the left by a distance q_3 , while the other moves up to the left by the same distance. In the meantime, the oxygen must move to the right by a distance $(2mq_3 \sin \theta)/M$, in order to keep the centre of mass unmoved. We are going to try to write down the kinetic and potential energies in terms of the internal coordinates q , q_2 and q_3 . It is easy to write down the kinetic energy in terms of the (x, y) coordinates:

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m(\dot{x}_3^2 + \dot{y}_3^2)$$

17.10.1 From geometry, we have: $y = c - c^3 \sin \theta$ $y_3 = c - c^3 \cos \theta$
17.10.2a,b $\dot{y}^2 = 2mc^3 \sin \theta \dot{\theta}$ $\dot{y}_3^2 = 2mc^3 \sin \theta \dot{\theta}$
17.10.3a,b $M \dot{x}^2 = -c^3 \sin \theta \dot{\theta}$
17.10.4a,b On putting these into equation 17.9.1 we obtain $T = mc^3 + m(1 + 2m/M)\dot{\theta}^2 + m\dot{\theta}^2 + (2m \sin^2 \theta / M)\dot{\theta}^2$ 17.10.5 For short, I am going to write this as $T = a_{\theta, \theta} + a_{\theta, \theta} \dot{\theta}^2$ 17.10.6 Now for the potential energy. The extension of the left hand spring is $q - q \sin \theta$ $q \cos \theta$ $2mq \cos \theta$ $2mq^3 \sin \theta \cos \theta = -q^3 + M = -q \sin \theta - q^2(1 + 2m/M) \cos \theta + q(1 + (2m \sin \theta \cos \theta)/M)$ 17.10.7

22 The extension of the right hand spring is $q_2 \sin \theta$ $q_2 \cos \theta$ $2mq_2 \cos \theta$ $2mq_2^3 \sin \theta$ $M = -q \sin \theta - q(1 + 2m/M) \cos \theta - q_2 \sin \theta + (2m \sin \theta / M)$ 17.10.8
The increase in the angle between the springs is 2θ $2q \cos \theta$ $2(1 + 2m/M)q^2 \sin \theta$ -- F F 17.10.9 The potential energy (above the equilibrium position) is $V = -\frac{1}{2} k(fq)^2 + -\frac{1}{2} k(fr)^2 + -\frac{1}{2} c(2\theta)^2$ 17.10.10 On substituting equations 17.10.7,8 and 9 into this, we obtain an equation of the form $V = b_{\theta, \theta} + 2b_{\theta, q}q + b_{\theta, q^2}q^2 + b_{\theta, q^3}q^3 + b_{\theta, M}M$ 17.10.11 where I leave it to the reader, if s/he wishes, to work out the detailed expressions for the coefficients. We still have a cross term, so we can't completely separate the coordinates, but we can easily apply Lagrange's equation to equations 17.10.6 and 11, and then seek simple harmonic solutions in the usual way. Setting the determinant of the coefficients to zero leads to the following equation for the angular frequencies of the normal modes: $b_{\theta, \theta} - (\omega^2 a_{\theta, \theta} b_{\theta, q}^2 - \omega^2 b_{\theta, q}^2 - \omega^2 b_{\theta, q^2}^2 - \omega^2 b_{\theta, q^3}^2 - \omega^2 b_{\theta, M}^2 = 0$ 17.10.12 Thus, given the masses and r , θ , k and c , one can predict the frequencies of the normal modes. Can one calculate k and c given the frequencies? I don't know, to tell the truth. Can I leave it to the reader to investigate further?

23 17.11 .4 General Vibrating System In a more general system, one with n

degrees of freedom and hence described by n generalized coordinates, the expressions for the kinetic and potential energies will be of the form $T = \frac{1}{2} \dot{q}_1^2 + \dots + \frac{1}{2} \dot{q}_n^2$, 17.11.1 and $V = \frac{1}{2} k_1 q_1^2 + \dots + \frac{1}{2} k_n q_n^2$, 17.11.2 On applying Lagrange's equation to q , we obtain $\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = 0$, 17.11.3 On seeking a simple harmonic solution, we obtain $(-m\omega^2)q + (k_1 q_1 + k_2 q_2 + \dots + k_n q_n) = 0$, 17.11.4 We obtain equations similar to equation 17.11.3 when applying Lagrange's equations to the other coordinates. Thus, for example, if there were three coordinates, the differential equations would be $m_1 \ddot{q}_1 + k_{11} q_1 + k_{12} q_2 + k_{13} q_3 = 0$, 17.11.5 This can be written for short (and for n coordinates, not just three) $\mathbf{M} \ddot{\mathbf{q}} + \mathbf{K} \mathbf{q} = 0$, 17.11.6 where $\mathbf{M}$ and $\mathbf{K}$ are $n \times n$ second-rank tensors.

24 17.12 A Driven System It would probably be useful before reading this and the next section to review Chapters 11 and 12. I k_1 m_1 k_2 m_2
Equilibrium: , x_1 , x_2 Displaced:  : $F = \sin \cot$ FIGURE XVII. 12
Figure XVII. 12 shows the same system as figure XVII.2, except that, instead of being left to vibrate on its own, the second mass is subject to a periodic force $F = \sin \cot$. For the time being, we'll suppose that there is no damping. Either way, it is not a conservative force, and Lagrange's equation will be used in the form of equation 13.4.12. As in section 17.2, the kinetic energy is $T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$. 17.12.1 Lagrange's equations are $\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_i} \right) - \frac{\partial T}{\partial x_i} + \frac{\partial V}{\partial x_i} = Q_i$, 17.12.2 $\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) - \frac{\partial T}{\partial x_1} + \frac{\partial V}{\partial x_1} = 0$ and $\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) - \frac{\partial T}{\partial x_2} + \frac{\partial V}{\partial x_2} = F \sin \cot$. 17.12.3 $\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) - \frac{\partial T}{\partial x_2} + \frac{\partial V}{\partial x_2} = F \sin \cot$ We have to identify the generalized forces P_1 and P_2 . In the nonequilibrium position, the extension of the left hand spring is x_1 and so the tension in that spring is $f_1 = k_1 x_1$. The extension of the right hand spring is $x_2 - x_1$ and so the tension in that spring is $f_2 = k_2 (x_2 - x_1)$. If x_1 were to increase by δx_1 , the work done would be $(f_1 - f_2) \delta x_1$, and therefore the generalized force associated with the coordinate x_1 is $P_1 = k_1 x_1 - k_2 (x_2 - x_1)$. If x_2 were to increase by δx_2 , the work done would be $(F - f_2) \delta x_2$, and therefore the generalized force associated with the coordinate x_2 is $P_2 = F - k_2 (x_2 - x_1)$. The lagrangian equations of motion therefore become

25 $m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0$ and $m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = F \sin \cot$. Seek solutions of the form $x_1 = A_1 \cos \omega t$ and $x_2 = A_2 \cos \omega t$. $(k_1 + k_2 - m_1 \omega^2) A_1 - k_2 A_2 = 0$ and $-k_2 A_1 + (k_2 - m_2 \omega^2) A_2 = F \sin \cot$. 17.12.4 17.12.5 The equations become 17.12.6 17.12.7 We do not, of course, now equate the determinants

of the coefficients to zero (why not?!), but we can solve these equations to obtain $X_1 = k_2 / \sin \cot (k + k_2 - m\omega^2)(k_2 - m_2\omega^2) - k_2^2$ 17.12.8 $(k + k_2 - m\omega^2)/\sin \cot$ and $x_2 =$ 17.12.9 $(\omega^2 - \omega_0^2)$ The amplitudes of these motions (and how they vary with the forcing frequency ω) are $k_2 /$ 17.12.10 $m_2\omega^4 - (mk^2 + m_2k + m_2k^2)\omega^2 + kk^2 (k + k_2 - m\omega^2)/$ 17.12.11 and $2 = ' m_2\omega^4 - (mk^2 + m_2k + m_2k^2)\omega^2 + kk^2$ where I have re-written the denominators in the form of a quadratic expression in ω^2 . For illustration I draw, in figure XVII. 13, the amplitudes of the motion of m (continuous curve, in black) and of m_2 (dashed curve, in blue) for the following data: $\omega = 1$, $k = k_2 = 1$, $m = 3$, $m_2 = 2$, when the equations become $1 \ 1 \ 60) \ 4 \ -- \ 70) \ 2 \ + \ 1 \ (60 \ 2 - 1)(0 \ 2 - 1)$ 17.12.12

26 and 2 $' -m''^2 / m \ 1 \ 1 \ -1 \ -2 \ -3 \ 2 \ - \ 300 \ 2 \ 6004 \ - \ 700 \ 2 \ + \ 1 \ 2 \ - \ 300 \ 2 \ (600 \ 2 - 1)(00 \ 2 - 1)$ FIGURE XVII. 1 3 I I , , ; I / .I 0.5 -4 0 1.5 ; / ω 17.12.13 Where the amplitude is negative, the oscillations are out of phase with the force F . The amplitudes go to infinity (remember we are assuming here zero damping) at the two frequencies where the denominators of equations 17.12.10 and 17.12.11 are zero. The amplitude of the motion of m_2 is zero when the numerator of equation 17.12.11 is zero. This is at an angular frequency of $\omega / (k + k_2) / m$, which is just the angular frequency of the motion of m_1 held by the two springs between two fixed points. 1 7.1 3 A Damped Driven System I'll leave the reader to add some damping to the system described in section 17.12. Let us here try it with the system described in section 17.7. We'll apply a periodic force to the left hand mass, and we'll suppose that the damping constant for each mass is $\gamma = b/m$. We could write the periodic force as $F = P \sin \cot$, but the algebra will be easier if we write it as $F = e^{i\omega t}$. If the initial condition is such that $F = 0$ when $t = 0$, then we choose just the imaginary part of this and subsequent expressions. The equations of motion are

27 and - the damping force b : - the tension in the left hand spring k_1x + the force F + the tension in the middle spring $k_2(y - x)$ (this last is a thrust whenever $y < x$) $m) = -$ the damping force b) - the thrust in the right hand spring k_1x + the tension in the middle spring $k_2(y - x)$ $\sin \cot k_2 m \ k_1 \ 1 \ > x$ $\rightarrow y$ FIGURE XVII. 14 That is, $m\ddot{x} + \gamma\dot{x} + (k_1 + k_2)x - k_2y = F e^{i\omega t}$ and $m_2\ddot{y} + \gamma_2\dot{y} + (k_2 + k_3)y - k_2x = 0$. 17.13.2 17.13.1 For the steady-state motion, seek solutions of the form $x = X e^{i\omega t}$, $y = Y e^{i\omega t}$; SO that $\dot{x} = i\omega X$ and $\dot{y} = i\omega Y$.

The equations then become $(k + k_2 - m\omega^2 + ibc\omega)x - k_2y = F e^{i\omega t}$ 17.13.3 and $-k_2x + (k + k_2 - m\omega^2 + ibc\omega)y = 0$. 17.13.4 There is now a little algebra to be carried out. Solve these equations for x and y , and when, in doing so, there is a complex number in the denominator, multiply top and bottom by the conjugate in the usual way, so as to get x and y in the forms $x' + ix''$ and $y' + iy''$. Then find expressions for the amplitudes and phases. After some algebra, the amount of which depends on one's skill, experience and luck (it is not always obvious how to gather terms in the most economical way, and you need some luck in this) you eventually get, for the amplitudes of the motion

$$x = \frac{F}{(k + k_2 - m\omega^2)^2 + b^2c^2\omega^2} [(k_2 - m\omega^2) \cos \phi + bc\omega \sin \phi]$$
 17.13.5
$$y = \frac{F}{(k + k_2 - m\omega^2)^2 + b^2c^2\omega^2} [bc\omega \cos \phi + (k_2 - m\omega^2) \sin \phi]$$
 17.13.6 There are many variables in these expressions, but in order to see qualitatively what the steady state motion is like, I'm going to put $m = 1$ and $k = 1$. I think if I also put $b = 1$, this will give light damping in the sense described in Chapter 11. As for k_2 , I am going to introduce a coupling coefficient c defined by $c = k_2 / (k + k_2)$ or $k_2 = ck$. This coupling constant will be close to zero if the middle spring is very weak, and 1 if the middle connector is a rigid rod. The equations now become $(1 - \omega^2 + ic\omega)x - cy = F e^{i\omega t}$ 17.13.7
$$-cy + (1 - \omega^2 + ic\omega)y = 0$$
 17.13.8 I dare say these expressions can be simplified, but it is late and my energy is flagging, and in any case I am not at all sure that "simplifying" them will much increase the computational efficiency. Figure XVII. 15 shows the amplitudes of the motions of the two masses as a function of frequency, for $c = 0.1, 0.5$ and 0.9 . The continuous black curves are for the left hand mass; the dashed blue curve is for the right hand mass.

FIGURE XVII.15 Amplitudes of the motions of the two masses as a function of frequency, for $c = 0.1, 0.5$ and 0.9 . The continuous black curves are for the left hand mass; the dashed blue curve is for the right hand mass.

CHAPTER 18 THE CATENARY 18.1 Introduction If a flexible chain or rope is loosely hung between two fixed points, it hangs in a curve that looks a little like a parabola, but in fact is not quite a parabola; it is a curve called a catenary, which is a word derived from the Latin catena, a chain. 18.2 The Intrinsic Equation to the Catenary FIGURE XVIII. 1 We consider the equilibrium of the portion AP of the chain, A being the lowest point of the

chain. See figure XVIII. 1 It is in equilibrium under the action of three forces: The horizontal tension T_0 at A; the tension T at P, which makes an angle with the horizontal; and the weight of the portion AP. If the mass per unit length of the chain is p and the length of the portion AP is s , the weight is psg . It may be noted that these three forces act through a single point. Clearly, and from which and $T_0 = T \cos t$ 18.2.1 $psg = T \sin t$, 18.2.2 $(\frac{s}{a})^2 + \frac{T_0^2}{T^2} = 1$ 18.2.3 $\tan t = \frac{psg}{T_0}$ 18.2.4

Introduce a constant a having the dimensions of length defined by $a = \frac{T_0}{pg}$. Then equations 18.2.3 and 4 become $T = [pg]s^2 + a^2$ and $s = a \tan t$. Equation 18.2.7 is the intrinsic equation (i.e. the s , equation) of the catenary. 18.2.5 18.2.6 18.2.7 18.3 Equation of the Catenary in Rectangular Coordinates, and Other Simple Relations The slope at some point is $y' = \frac{dy}{dx} = \tan t = S$, from which $ds = \frac{dy}{y'}$ - But, from the usual $dx^2 + dy^2 = ds^2$ pythagorean relation between intrinsic and rectangular coordinates $ds = \frac{1}{\sqrt{1 + y'^2}} dx$, this becomes $(1 + y'^2) = a^2 y'^2$ - On integration, with the condition that $y' = 0$ where $x = 0$, this becomes $y' = \sinh(x/a)$, $y = a \cosh(x/a) + C$. 18.3.1 C. 18.3.3 If we fix the origin of coordinates so that the lowest point of the catenary is at a height a above the x -axis, this becomes $y = a \cosh(x/a)$ 18.3.4 This, then, is the x, y equation to the catenary. The x -axis is the directrix of this catenary. The following additional simple relations are easily derived and are left to the reader: $s = a \sinh(x/a)$, 18.3.5 $y^2 = a^2 + s^2$ 18.3.6 $y = a \sec t$. 18.3.7 and, on further integration, 18.3.2

$x = a \ln(\sec t + \tan t)$, 18.3.8 $T = [pg]y^2$. 18.3.9 Equations 18.3.7 and 8 may be regarded as parametric equations to the catenary. If one end of the chain is fixed, and the other is looped over a smooth peg, equation 18.3.9 shows that the loosely hanging vertical portion of the chain just reaches the directrix of the catenary, and the tension at the peg is equal to the weight of the vertical portion. Exercise: By expanding equation 18.3.4 as far as x^2 , show that, near the bottom of the catenary, or for a tightly stretched catenary with a small sag, the curve is approximately a parabola. Actually, it doesn't matter what equation 18.3.4 is - if you expand it as far as x^2 , provided the x^2 term is not zero, you'll get a parabola - so, in order not to let you off so lightly, show that the semi latus rectum of the parabola is a . Exercise: Expand equation 18.3.5 as far as x^3 . Now: let $2s =$ total length of chain, $2k =$ total span, and $d =$ sag. Show that for a shallow catenary $8s^2 - k^2 = 6a^3$ and $k^2 = 2ad$, and

hence that length - span = $\frac{1}{2} \times \text{sag}/\text{span}$. Question: A cord is stretched between points on the same horizontal level. How large a force must be applied so that the cord is no longer a catenary, but is accurately a straight line? Answer: There is no force however great Can stretch a string however fine Into a horizontal line That shall be accurately straight. And if anyone can find the origin of that fine piece of doggerel, please let me know. And here's something for engineers. We, the general public, expect engineers to build safe bridges for us. The suspension chain of a suspension bridge, though scarcely shallow, is closer to a parabola than to a catenary. There is a reason for this. Discuss. 18.4 Area of a Catenoid A theorem from the branch of mathematics known as the calculus of variations is as follows. Let $y = y(x)$ with $y' = dy/dx$, and let $f(y, y', x)$ be some function of y , y' and x . Consider the line integral off from A to B along the route $y = y(x)$.

4 18.4.1 In general, and unless f is a function of x and y alone, and not off, the value of this integral will depend on the route (i.e. $y = y(x)$) over which this line integral is calculated. The theorem states that the integral is an extremum for a route that satisfies 18.4.2 By "extremum" we mean either a minimum or a maximum, or an inflection, though in many - perhaps most - cases of physical interest, it is a minimum. It can be difficult for a newcomer to this theorem to try to grasp exactly what this theorem means, so perhaps the best way to convey its meaning is to start by giving a simple example. Following that, I give an example involving the catenary. There will be another example, involving a famous problem in dynamics, in Chapter 19, and in fact we have already encountered an application of it in Chapter 14 in the discussion of Hamilton's variational principle. Let us consider, for example, the problem of calculating the distance, measured along some route $y(x)$ between two points; that is, we want to calculate the arc length l . From the usual pythagorean relation between ds , dx and dy , this is $(1+y'^2)^{1/2} dx$. The variational principle says that this distance - measured along $y(x)$ - is least for a route $y(x)$ that satisfies equation 18.4.2, in which in this case $f = (1 + y'^2)^{1/2}$. For this case, we have $df/dy' = y'$ and $df/dy = 0$. Thus integration of equation 18.4.2 gives $y' = c(1 + y'^2)^{-1/2}$, 18.4.3 c where c is the integration constant. If we solve this for y' , we obtain $y' = \frac{c}{\sqrt{1 - c^2}}$, which is just another constant, which I'll write as a , so that $y' = a$. Integrate this to find $y = ax + b$. 18.4.4 This probably seems rather a long way to prove that the shortest distance between two points is a straight line - but that wasn't the point of the exercise.

The aim was merely to understand the meaning of the variational principle. Let's try another example, in which the answer will not be so obvious. Consider some curve $y = y(x)$, and let us rotate the curve through an angle (which need not necessarily be a full 2π radians) about the y -axis. An element ds of the curve can be written as

$\sqrt{1 + y'^2} dx$, and the distance moved by the element ds (which is at a distance x from the y -axis) during the rotation is θx . Thus the area swept out by the curve is $\frac{1}{2} \int_0^L \theta x^2 \sqrt{1 + y'^2} dx$. For what shape of curve, $y = y(x)$, is this area least? The answer is - a curve that satisfies equation 18.4.2, where $f = \frac{1}{2} x^2 \sqrt{1 + y'^2}$. For this function, we have $f_x = x \sqrt{1 + y'^2}$ and $f_{y'} = xy'$. Therefore the required curve satisfies $J + y' \frac{dy}{dx} = 0$. 18.4.6 That is, $\frac{dy}{dx} = -\frac{x}{y}$. 18.4.7 On substitution of $x = a \cosh u$, and looking up everything we have forgotten about hyperbolic functions, and integrating, we obtain $y = a \cosh(x/a)$. 18.4.8 Thus the required curve is a catenary. If a soap bubble is formed between two identical horizontal rings, one beneath the other, it will take up the shape of least area, namely a catenoid.

19.1 Introduction CHAPTER IXX THE CYCLOID 1.5 >, 1 FIGURE I)0(. 1 0.5 1 1.5 2 A 2.5 3 x Let us set up a coordinate system Oxy , and a horizontal straight line $y = 2a$. We imagine a circle of diameter $2a$ between the x -axis and the line $y = 2a$, and initially the lowest point on the circle, P , coincides with the origin of coordinates O . We now allow the circle to roll counterclockwise without slipping on the line $y = 2a$, so that the centre of the circle moves to the right. As the circle rolls on the line, the point P describes a curve, which is known as a cycloid. When the circle has rolled through an angle 2θ , the centre of the circle has moved to the right by a horizontal distance $2a\theta$, while the horizontal distance of the point P from the centre of the circle is $a \sin 2\theta$, and the vertical distance of the point P below the centre of the circle is $a \cos 2\theta$. Thus the coordinates of the point P are $x = a(2\theta + \sin 2\theta)$, 19.1.1 and $y = a(1 - \cos 2\theta)$. 19.1.2 These are the parametric equations of the cycloid. Equation 19.1.2 can also be written $y = 2a \sin^2 \theta$. 19.1.3

2 Exercise: When the x -coordinate of P is $2.500a$, what (to four significant figures) is its y - coordinate? Solution: We have to find 2θ by solution of $2\theta + \sin 2\theta = 2.5$. By Newton-Raphson iteration (see Chapter 1 of the Stellar Atmospheres notes in this series) or otherwise, we find $2\theta = 0.931\ 599\ 201$

radians, and hence $y = 0.9316a$. 19.2 Tangent to the Cycloid The slope of the tangent to the cycloid at P is dy/dx , which is equal to $(dy/d\theta)/(dx/d\theta)$, and these can be obtained from equations 19.1.1 and 19.1.2. Exercise: Show that the slope of the tangent at P is $\tan \theta$. That is to say, the tangent at P makes an angle θ with the horizontal. Having done that, now consider the following: Let A be the lowest point of the circle. The angle that AP makes with the horizontal is given by $\tan^{-1} x/(2a - y)$. Exercise: Show that $\theta = \tan^{-1} x/(2a - y)$. Therefore the line AP is the tangent to the cycloid at P; or the tangent at P is the line AP. 19.3 The Intrinsic Equation to the Cycloid An element ds of arc length, in terms of dx and dy , is given by the theorem of Pythagoras: $ds^2 = dx^2 + dy^2$, or, since x and y are given by the parametric equations 19.1.1 and 19.1.2, by $ds = a d\theta$. And of course we have just shown that the intrinsic coordinate (i.e. the angle that the tangent to the cycloid makes with the horizontal) is equal to θ . Exercise: Integrate ds (with initial condition $s = 0$, cycloid is $s = 0$ using $\theta = 0$) to show that the intrinsic equation to the cycloid is $s = 4a\theta$. 19.3.1 Also, eliminate θ from equations 19.3.1 and 19.1.2 to show that the following relation holds between arc length and height on the cycloid: $2s = 4ay$. 19.3.2

19.4 Variations In sections 19.1, 2, 3, we imagined that the cycloid was generated by a circle that was rolling counterclockwise along the line $y = 2a$. We can also imagine variations such as the circle rolling clockwise along $y = 0$, or we can start with P at the top of the circle rather than at the bottom. I summarise in this section four variations. The distinction between θ and ϕ is as follows. The angle that the tangent to the cycloid makes with the positively-directed x-axis is ϕ ; that is to say, $dy/dx = \tan \phi$. The circle rolls through an angle 2θ . There is a simple relation between θ and ϕ , which is different for each case. In each figure, x and y are plotted in units of a . The vertical height between vertices and cusps is $2a$, the horizontal distance between a cusp and the next vertex is $2a$, and the arc length between a cusp and the next vertex is $4a$. at the origin. Figure IXX.2. Circle rolls counterclockwise along $y = 2a$. P starts at the bottom. The cusps are up. A vertex is $x = a(2\theta + \sin 2\theta)$ 19.4.1 $y = 2a \sin^2 \theta$ 19.4.2 $s = 4a \sin \theta$ 19.4.3 $s = 8a \sin^2 \theta$ 19.4.4 $\phi = 0$. 19.4.5 $2\theta > \phi$ 19.4.6 $\phi = 0$ FIGURE IXX.2 2 4 6 8 10 12 x

4 II. Circle rolls clockwise along $y = 0$. origin. Figure IXX. 3. P starts at the bottom. The cusps are down. A cusp is at the $x = a(2\theta - \sin 2\theta)$ 19.4.6 $y = 2a \sin^2 \theta$ 19.4.7 $s = 4a(1 - \cos \theta)$ 19.4.8 $s = 8a(1 - \cos^2 \theta)$ 19.4.9 $\phi = 90^\circ - \theta$.

19.4.10 2 0 2 FIGURE I)0(.3 i i i 4 6 8 10 12 x

III. Circle rolls clockwise along $y = 0$. P starts at the top. Figure IXX.4. $a(20 + \sin 2\theta) - 2a \cos 2\theta - 4a \sin \theta - 8a(2a - y) - 180^\circ - 0$. The cusps are down. A vertex is at $x = 0$. 19.4.11 19.4.12 19.4.13 19.4.14 19.4.15 2 >,1 FIGURE IXX. 4 2 4 6 8 10 12

IV. Circle rolls counterclockwise along $y = 2a$. P starts at the top. The cusps are up. A cusp is at $X:0$. Figure IXX. 5. $x = a(20 - \sin 2\theta)$ 19.4.16 $y = 2a \cos 2\theta$ 19.4.17 $s = 4a(1 - \cos \theta)$ 19.4.18 $s - 8as + 8a(2a - y) = 0$ 19.4.19 $= 90^\circ + \theta$. 19.4.20 2 >,1 FIGURE IXX. 5 2 4 6 8 10 12 14 x

7 19.5 Motion on a Cycloid, Cusps Up We shall imagine either a particle sliding down the inside of a smooth cycloidal bowl, or a bead sliding down a smooth cycloidal wire, figure IXX.6. 1.5 >, 1 0.5 0 0 FIGURE IXX. 6 I I I I I 0.5 1 1.5 2 2.5 3 We shall work in intrinsic coordinates to obtain the tangential and normal equations of motion. (For a reminder of the use of intrinsic coordinates, they were used briefly, for example, in Section 7.3) These equations are, respectively: $-g \sin \theta$ 19.5.1 and $m w^2 - R - mg \cos \theta$. 19.5.2 P Here R is the normal (and only) reaction of the bowl or wire on the particle and p is the radius of curvature. The radius of curvature is $ds/d\theta$, which, from equation 19.3.1, (or equations 19.4.3 and 19.4.5) is $p = 4a \cos \theta$. 19.5.3

From equations 19.3.1 and 19.5.1 we see that the tangential equation of motion can be written, without approximation: $-g \sin \theta$ 19.5.4 $4a$ This is simple harmonic motion of period $4\pi/\sqrt{g}$, independent of the amplitude of the motion. This is the isochronous property of the cycloid. Likewise, if the particle is released from rest, it will reach the bottom of the cycloid in a time $\pi/\sqrt{g}$, whatever the starting position. Let us see if we can find the value of R where the generating angle is θ . Let us suppose that the particle is released from rest at a height y_0 above the x -axis (generating angle $= 0$); what is its speed v when it has reached a height y (generating angle θ)? Clearly this is given by $mt^2 = mg(y_0 - y)$, 19.5.5 and, following equation 19.3.2, and recalling that $0 = \theta$, this is $v^2 = 2ga(\cos^2 \theta - \cos 2\theta)$. 19.5.6 On substituting this and equation 19.5.3 into equation 19.5.2, we find for R : $R = mg(1 + 2\cos^2 \theta - \cos 2\theta)$. 19.5.7 $2 \cos \theta$ 19.6 Motion on a Cycloid, Cusps Down We imagine a particle sliding down the outside of an inverted smooth cycloidal bowl, or a

bead sliding down a smooth cycloidal wire. We shall suppose that, at time $t = 0$, the particle was at the top of the cycloid and was projected forward with a horizontal velocity v_0 . See figure IXX.7. This time, the equations of motion are $\ddot{s} = g \sin \theta$ and $m v^2 / r = mg \cos \theta - R$. 19.6.2 P By arguments similar to those made in Section 9.5, we find that $\dot{s} = gs$. 19.6.3 4a

FIGURE IXX. 7 The general solution to this is where With the initial condition given (at $t = 0$, $S = Ae^{pt} + Be^{-pt}$), we can find A and B and hence: $S = D \cosh(pt)$. Again proceeding as in Section 19.5, we find for R: $R = 4a \cos^2 \theta (4g \cos^2 \theta - v_0^2 / (2a))$. 19.6.7 So - what happens?

10 If the constraint is two-sided (bead sliding on a wire) R becomes zero when $\cos^2 \theta = v_0^2 / (2ga)$, and thereafter R is in the opposite direction. If the constraint is one-sided (particle sliding down the outside of a smooth cycloidal bowl): 1. If $v_0 > \sqrt{4ga}$, the particle loses contact at the moment of projection. 2 If $v_0 < \sqrt{4ga}$, the particle loses contact as soon as $\cos^2 \theta = v_0^2 / (2ga)$. If v_0 is very small (i.e. very much smaller than $\sqrt{2x}$), this will happen when $\theta = 45^\circ$; for faster initial speeds, contact is lost sooner. Example. A particle is projected horizontally with speed $v_0 = 1 \text{ m/s}$ from the vertex of the smooth cycloidal hill $x = a(20 + \sin 20)$ $y = 2a \cos^2 \theta$, where $a = 2 \text{ m}$.

Assuming that $g = 9.8 \text{ m/s}^2$, how long does it take to get halfway down the hill (i.e. to $y = a$)? We have to use equation 19.6.6. With the numerical data given, this is $s = 0.451754(e^{1.565248t} - e^{-1.565248t})$. We can find s from equation 19.4.12, which gives us $s = 2.828427 \text{ m}$. now have to solve $6.26099 = -1/k$, or $-6.26099 - 1 = 0$. hence $t = 1.19 \text{ s}$. If we let $t = e^{1.565248t}$ we From this, $t = 6.41683$ and I leave it to the reader to calculate R at this time - and indeed to see whether the particle loses contact with the hill before then. Perhaps the fact that I got a positive real root for means that we are all right and the particle is still in contact - but I wouldn't be sure of that. I leave it to the reader to investigate further. 19.7 The Brachystochrone Property of the Cycloid A small point. The word is sometimes spelled brachistochrone, and I have no recommendation one way or the other. For what it's worth, the only dictionary within easy reach of my desk has brachiopod and brachycephalic. In any case, the word is derived from Greek, and means shortest time. The famous brachystochrone problem is this: A smooth wire, which can be of any desired length, is to connect two points O and P; P is at a lower

level than O, but is not vertically below O. The wire is to be bent to a shape, and cut to a length, such that the time taken for a bead to slide down the wire from O to P is least.

11 It is not easy to prove that the required curve is a cusps-up cycloid; but it is quite reasonable to speculate or to guess that this might be so. And, having speculated that it might be a cycloid, it is easy to verify that the required curve is indeed a cusps-up cycloid, the bead starting from rest at a cusp of the cycloid. A speculation might go something like this. Generally one would expect that the further P is from O, the longer it will take for the bead to slide from O to P. But, if O and P are connected with a cycloidal wire, the time taken to go from O to P does not increase with distance. (See the isochronous property of the cycloid discussed in Section 19.5.) Thus, as you increase the distance between O and P, the time taken to travel by any route other than the cycloidal one must take longer than the cycloidal one. This argument may not sound like a rigorous proof, though it is enough to arouse our suspicions and to test whether it is correct. Since I am going to deal with a bead sliding downwards under gravity, I am going to find it convenient to set up our coordinate axes such that x increases to the right, and y increases downwards. In that case, the parametric equations to a cusps-up cycloid, with the origin at a cusp, are $x = a(2\theta - \sin 2\theta)$, 19.7.1 and $y = 2a \sin^2 \theta$ 19.7.2 - and these are the equations that we shall be testing. The time taken for the bead to travel a distance ds along the wire, while it is moving at speed v is ds/v . In (x, y) coordinates, ds is $\sqrt{dx^2 + dy^2}$, where $y' = dy/dx$. Also, the speed reached is related (by equating the gain in kinetic energy to the loss of potential energy) to the vertical distance y dropped by $v = \sqrt{2gy}$. Thus the time taken to go from O to P is $\int \frac{ds}{v} = \int \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gy}}$ 19.7.3 This is least (see Chapter 18 for a discussion of this theorem from the calculus of variations) for a function $y(x)$ that satisfies $\frac{d}{dx} \left(\frac{y'}{\sqrt{y}} \right) = 0$ 19.7.4 We have: $f = 41 + x/7$ 19.7.5

12 $y = 2y^{3/2}$, 19.7.6 and $af Y' = 19.7.7$ It is left for the reader to see whether equations 19.7.1 and 2 satisfy equation 19.7.4. You should find that both sides of the equation are equal to $-1/(4x/a^{3/2} \sin^4 \theta)$ Thus our speculation is confirmed, and a cusps-up cycloid is indeed the curve that offers passage from O to P in the shortest time. 19.8 Contracted and Extended Cycloids As in Section 19.1, we consider a circle of radius a rolling to the right on the line $y = 2a$. The point P is initially below the centre of the circle, but, instead

of being on the rim of the circle, its distance from the centre of the circle is r . If $r < a$, the path described by P will be a contracted cycloid; if $r > a$, the path is an extended cycloid. (I think there's a case for using this nomenclature the other way round, but most authors seem to use "contracted" for $r < a$ and "extended" for $r > a$.) It should not take long to be convinced, by arguments similar to those in Section 19.1, that the parametric equations to a contracted or extended cycloid are $x = 2a\theta + r \sin 2\theta$ 19.8.1 and $y = a - r \cos 2\theta$. 19.8.2 These are illustrated in figures IXX.8 and IXX.9 for a contracted cycloid with $r = 0.5a$ and an extended cycloid with $r = 1.5a$.

FIGURE IXX.8 0 2 4 6 8 10 12 x 14 FIGURE IXX.9 2 >, 1 0 -1 0 1 1 1 1 1 2 4 6 8 10 12 14

14 19.9 The Cycloidal Pendulum 2 1.5 1 0.5 0 >, -0.5 -1 -1.5 -2 -2.5 -3
FIGURE IXX. 10 1 1 1 1 1 1 -3 -2 -1 0 1 2 3 x Let us imagine building a wooden construction in the shape of the cycloid $x = a(2\theta - \sin 2\theta)$ 19.9.1 $y = 2a \cos^2 \theta$ 19.9.2 shown with the thick line in figure IXX. 10. Now suspend a pendulum of length $4a$ from the cusp, and allow it to swing to and fro, partially wrapping itself against the wooden frame as it does so. If the arc length from the cusp to P is s , then the length of the "free" string is $4a - s$, and so the coordinates of the bob at the end of the pendulum are $x = a(2\theta - \sin 2\theta) + (4a - s) \sin \theta$ and $y = 2a \cos^2 \theta - (4a - s) \cos \theta$. 19.9.3 19.9.4

15 (You will need to remind yourself of the exact meaning of t and also make use of equation 19.4.20.) Now equation 19.4.18 tells us that $s = 4a(1 - \cos \theta)$, and, on substitution of this in equations 19.9.3 and 4, we find (after a very little algebra and trigonometry) for the parametric equations to the path described by the bob of the pendulum: $x = a(2\theta + \sin 2\theta)$ 19.9.5 and $y = -2a \cos^2 \theta$. 19.9.6 Thus the path of the pendulum bob (shown as a dashed line in figure IXX. 10) is a cycloid, and hence its period is independent of its amplitude. (Recall Section 19.5.) Thus the pendulum is isochronous or tautochronous. It is astonishing to learn that Huygens constructed just such a pendulum as long ago as 1673. 19.10 Examples of Cycloidal Motion in Physics Several examples of cycloidal motion in physics come to mind. One is the nutation of a top, which is described in Section 4.10 of Chapter 10. Earth's axis nutates in a similar fashion. Another well known example is the

motion of an electron in crossed electric and magnetic fields. This is described in Chapter 8 of the Electricity and Magnetism section of these notes. In cosmology, if the mean density of the Universe is low, the Universe expands indefinitely, but, if the density is higher than a certain critical density, the (dimensionless) scale factor R of the Universe expands and contracts with time t according to the following parametric cycloidal equations: $R = \frac{F}{2} (1 - \cos 2\theta)$, $19.10.1$ $\frac{2}{3} (r_0 - 1) \theta = \frac{2}{3} (C/H_0 - 1) \frac{3}{2} (2\theta - \sin 2\theta)$. $19.10.2$ Here t is expressed in units of the reciprocal of the present Hubble constant, and C/H_0 is the ratio of the present density of the Universe to the density required to "close" the Universe. A less well known example concerns the propagation of sound in the atmosphere. In the troposphere, which is the lower part of the atmosphere up to about 11 km, the temperature decreases roughly linearly with height. In that case sound travels through the troposphere in a cycloidal path. The speed of sound in a gas is proportional to the square root of the temperature. (If you are wondering how it depends on pressure P and density ρ , the answer is that it depends on the ratio P/ρ - and this ratio is proportional to the temperature.) In any case, if the temperature decreases linearly with height, the sound speed v varies with height y as $v = v_0 \sqrt{1 - cy}$, $19.10.3$

16 where c is a constant, equal to about 0.023 km^{-1} . Now to trace a sound ray through the atmosphere, we have to understand how the direction of propagation changes as the sound passes through layers of air of different temperature. This is governed, as with light, by Snell's law (see figure IXX. 11): $du \sin \theta = -v \tan \theta d\theta$. $19.10.4$  FIGURE IXX. 11 Snell's law states that when sound (or light) enters a slower medium (i.e. one in which the speed of propagation is slower) it is bent towards the normal. I have drawn figure IXX. 11 to represent the situation in the troposphere where the temperature (and hence the sound speed v) decreases with height. That is, dv/dy is negative. In other words dv in figure IXX is negative, and equation 19.10.4 indicates that dv is positive, as drawn. In case you do not recognize this differential form of Snell's law, try integrating it from v to v_2 and from V to V_2 , and it should assume its more familiar integral form. If you now eliminate v between equations 19.10.3 and 4, you will get a differential relation between y and t , which, upon integration, becomes $cy = 1 - \cos^2 \theta$. $19.10.5$ $\cos^2 \theta$ where θ_0 is the ground-level value of θ . If we introduce $1 - \cos^2 \theta = \sin^2 \theta$ $19.10.6$

17 equation 19.10.5 can be conveniently re-written $y = 2a(\sin 2t - \sin 2t_0) = 2a(\cos 2t_0 - \cos 2t)$. 19.10.7 Now $\tan t = dy/dx$, and elimination of y between this and equation 19.10.7 will give a differential relation between x and t , which, upon integration, becomes $x = a[2(t - t_0) + \sin 2t - \sin 2t_0]$. 19.10.8 Equations 19.10.7 and 19.10.8 are the parametric equations of the sound path through the troposphere, and describe a cycloid. Problem for a Rainy Day: If $x = 2.0$ and $y = 1.6$, what are t and t_0 ? make it = $69^\circ 17'$, $t_0 = 15^\circ 52'$.

APPENDIX A Miscellaneous Problems In this Appendix I offer a number of random problems in classical mechanics. They are not in any particular order - they come just as I happen to think of them, and they are not necessarily related to any of the topics discussed in any of the chapters. They are intended just to occupy you on those dull, rainy days when you have nothing better to do. Solutions will be in Appendix B - except that whenever I add any new problems to Appendix A, which I shall from time to time, I shall wait a few days before posting the solutions in Appendix B. 1. No book on classical mechanics is complete without a problem of a ladder leaning against a wall. Here, then, is a ladder problem - except that it is nothing whatever to do with mechanics, and it is put here just for fun. It is a problem only in geometry, yet it is one which some people at first find difficult. It even seems difficult to try to find an approximate solution by trying to draw it accurately to scale, and I have deliberately not drawn it to scale, so you can't find the answer merely by taking a ruler and measuring it! Two ladders, of lengths 8 m and 10 m, are leaning against two walls as shown. Their point of intersection is 3 m above the ground. What is the distance between the walls?

A pendulum of length l_0 , is set into motion so that it describes a cone as shown of semi-vertical angle θ , the bob describing a horizontal circle at angular speed ω . Show that $\cos \theta = g / l_0 \omega^2$. This, of course, is a very trivial problem not worthy of your mettle. It is given only as an introduction to the next problem. 3. (a) A string of a pendulum passes through a board as shown in the figure below, in such a manner that, by lowering or raising the board, the length of the string below the board can be varied. The part below the board is initially of length l_0 , and it is set into motion as a conical pendulum so that the angular speed and the semi vertical angle are related by $g \cos \theta = l_0 \omega^2$. As the board is raised or lowered (or alternatively the pendulum is lowered or raised) and consequently the length l below the board is varied,

the semivertical angle θ will change and so will the angular speed ω . (The symbols θ_0 , ω_0 and C_2 represent the initial values of these quantities.)

Show that i. $\frac{1}{3} \sin^3 \theta \tan \theta$ is constant, ii. $\frac{1}{\cot^2 \theta}$ is constant, iii. $\frac{1}{\tan^2 \theta}$ is constant. (b) Start with the following initial conditions: $\theta_0 = 50^\circ$, $\omega_0 = 5 \text{ rad/s}$ and assume that $g = 9.8 \text{ m/s}^2$, so that $\omega_0 = 38 \frac{1}{2} \text{ rev/s}$. i. Plot a graph of θ (vertically) versus l (horizontally), for $l = 0$ to 1 m . 40 cm, what is θ correct to one arcmin? ii. Plot a graph of ω (vertically) versus θ (horizontally), for $\theta = 0$ to 70° . iii. Plot a graph of ω (vertically) versus l (horizontally) for $l = 16 \text{ cm}$ to 1 m . = 60 cm, what is ω correct to four significant figures? When $l =$ When $l =$

The next few problems involve a rod with its lower end in contact with a horizontal table and the rod falling over from an initial vertical (or inclined) position. There are several versions of this problem. The table could be smooth, so that the rod freely slips over the table. Or the lower end could be freely hinged at the table, so that the lower end does not move as the rod falls over. Or the table might be rough, so that the rod might or might not slip. 4. A uniform rod of mass m and length $2l$ is initially vertical with its lower end in contact with a smooth horizontal table. It is given an infinitesimal angular displacement from its initial position, so that it falls over. When the rod makes an angle θ with the vertical, find: The angular speed of the rod; The speed at which the centre of the rod is falling; The speed at which the lower end of the rod is moving; The normal reaction of the table on the rod. Show that the speed of the lower end is greatest when $\theta = 37^\circ 50'$. If the length of the rod is 1 metre, and $g = 9.8 \text{ m/s}^2$, what is the angle θ when the speed of the lower end is 1 m/s ? 5. A uniform rod is initially vertical with its lower end smoothly hinged to a horizontal table. Show that, when the rod falls over, the reaction of the hinge upon the rod is vertical when the rod makes an angle $48^\circ 11'$ with the vertical, and is horizontal when the rod makes an angle $70^\circ 31'$ with the vertical. 6. A uniform rod of length 1 metre, with its lower end smoothly hinged to a horizontal table, is initially held at rest making an angle of 40° with the vertical. It is then released. If $g = 9.8 \text{ m/s}^2$, calculate its angular speed when it hits the table in a horizontal position (easy) and how long it takes to get there (not so easy). 7. A uniform rod is initially vertical with its lower end in contact with a rough horizontal table, the coefficient of friction being μ . Show that: a. If $\mu < 0.3706$, the lower end of the rod must slip

before the rod makes an angle θ with the vertical of $35^\circ 05'$. b. If it > 0.3706 , the rod will not slip before $\theta = 51^\circ 15'$, but it will certainly slip before $\theta = 70^\circ 31'$. If it $= 0.25$, at what angle θ will the lower end slip? If it $= 0.75$?

8. It is time for another ladder problem. Most ladders in elementary mechanics problems rest on a rough horizontal floor and lean against a smooth vertical wall. In this problem, both floor and wall are smooth. The ladder starts making an angle of θ_0 with the vertical, and then it is released. It immediately starts to slip, of course. After a while it will cease contact with the smooth vertical wall. Show that, at the moment when the upper end of the ladder loses contact with the wall, the angle θ that the ladder makes with the vertical is given by $\cos \theta = \cos \theta_0$. 9. If you managed that one all right, this one, which is somewhat similar, should be easy. Maybe. A uniform solid semicylinder of radius a and mass m is placed with its curved surface against a smooth vertical wall and a smooth horizontal floor, its base initially being vertical. It is then released. Find the reaction N of the floor on the semicylinder and the reaction N_2 of the wall on the semicylinder when its base makes an angle θ with the vertical. Show that the semicylinder loses contact with the wall when $\theta = 90^\circ$, and that it then continues to rotate until its base makes an angle of $39^\circ 46'$ with the vertical before it starts to fall back.

Many problems in elementary mechanics involve a body resting upon or sliding upon an inclined plane. It is time to try a few of these. The first one is very easy, just to get us started. The two following that might be more interesting. 10. A particle of mass m is placed on a plane which is inclined to the horizontal at an angle α that is greater than $\tan^{-1} \mu$, where μ is the coefficient of limiting static friction. What is the least force required to prevent the particle from sliding down the plane? 11. A cylinder of mass m , radius a , and rotational inertia $\frac{1}{2}ka^2$ rolls without slipping down the rough hypotenuse of a wedge of mass M , the smooth base of which is in contact with a smooth horizontal table. The hypotenuse makes an angle α with the horizontal, and the gravitational acceleration is g . Find the linear acceleration of the wedge as it slips along the surface of the table, in terms of m , M , g , a , k and α . [Note that by saying that the rotational inertia is $\frac{1}{2}ka^2$, I am letting the question apply to a hollow cylinder, or a solid cylinder, or even a hollow or

solid sphere.] 12. $x y$

A particle is placed on a rough plane inclined at an angle θ to the horizontal. It is initially in limiting static equilibrium. It is given an initial velocity V_0 along the x -axis. Ignoring the small difference between the coefficients of moving and limiting static friction, show that at a point on the subsequent trajectory where the tangent to the trajectory makes an angle with the x -axis, the speed V is given by $V = V_0(1 + \cos \theta)$. What is the limiting speed reached by the particle after a long time? 13. Calculate the moment of inertia of a hollow sphere, mass M , outer radius a , inner radius $x a$. Express your answer in the form $I = \frac{1}{2} M a^2 f(x)$. What does your expression become if $x = 0$? And if $x = 1$?

Calculate the moment of inertia of a spherical planet of outer radius a , consisting of a dense core of radius $x a$ surrounded by a mantle of density s times the density of the core. Express your answer in the form $I = \frac{1}{2} M a^2 f(x, s)$. Make sure that, if the density of the core is zero, your expression reduces to the answer you got for problem 13. Draw graphs of $I/(Ma^2)$ versus x (x going from 0 to 1), for s : 0.2, 0.4, 0.6 and 0.8. Show that, for a given mass M and density ratio s , the moment of inertia is least for a core size given by the solution of $2(1-s)x^5 + 15x^2 - 9 = 0$. For a mantle-to-density ratio of 0.6, calculate the core size for which the moment of inertia is least and calculate (in units of $\frac{1}{2} M a^2$) the moment of inertia for that core. Now let's see if we can determine the core size from a knowledge of the moment of inertia. It is sometimes asserted that one can determine the moment of inertia (and hence the core size) of a planet from the rate of precession of the orbit of a satellite. I am not sure how this would work with a planet such as Mercury, which has never had a satellite in orbit around it. (Mariner 10, while in orbit around the Sun, made three fly-bys past Mercury). Unless a planet departs from spherical symmetry, the orbit of a satellite will not precess, since the gravitational field is then identical with that from a point mass. And, even if a planet were dynamically oblate, the rate of precession allows us to determine the dynamical ellipticity $(C - A)/C$, but not either moment of inertia separately. Nevertheless, let's suppose that the moment of inertia of a planet is $(0.92 \pm 1\%) \times \frac{1}{2} M a^2$; specifically, let's suppose that the moment of inertia has been determined to be between 0.911 and $0.929 \times \frac{1}{2} M a^2$, and that the mantle-to-core density ratio is known (how?) to be 0.6. Calculate

the possible range in the value of the core radius x . 15. A rectangular brick of length $2l$ rests (with the sides of length $2l$ vertically) on a rough semicylindrical log of radius R . The drawing below shows three such bricks. In the first one, $2l$ is quite short, and it looks as if it is stable. In the second one, $2l$ is rather long, and the equilibrium looks decidedly wobbly. In the third one, we're not quite sure whether the equilibrium is stable or not. What is the longest brick that is stable against small angular displacements from the vertical?

10 16. A Thing with a semicylindrical (or hemispherical) base of radius a is balanced on top of a rough semicylinder (or hemisphere) of radius b as shown. The distance of the centre of mass of the Thing from the line (or point) of contact is l . Show that the equilibrium is stable if $1 + \frac{l}{a} + \frac{a}{b} = \frac{b}{l}$, is the equilibrium stable if the Thing is (1) A hollow semicylinder? (2) A hollow hemisphere? (3) A uniform solid semicylinder? (4) A uniform solid hemisphere?

11 17. A log of square cross-section, sides $2a$, rests on two smooth pegs a distance $2ac$ apart, one of the diagonals making an angle θ with the vertical. Show that, if $c < 1/\sqrt{2} = 0.707$ the only equilibrium position possible is $\theta = 90^\circ$, but that this position is unstable; consequently, following a small displacement, the logs will fall out of the pegs. Show that if the pegs are farther apart, with $c > 0.707$, three equilibrium positions are possible. Which of them are stable, and which are unstable? If $c = 1/\sqrt{2}$, what are the possible equilibrium values of θ ? 18.

12 A uniform solid hemisphere of radius a rests in limiting static equilibrium with its curved surface in contact with a smooth vertical wall and a rough horizontal floor (coefficient of limiting static friction μ). Show that the base of the hemisphere makes an angle θ with the floor, where $\sin \theta = \frac{3}{2\mu + 1}$. Calculate the value of θ if (a) $\mu = 1$ and (b) $\mu = 3$. What happens if $\mu > 3$? 19. A uniform rod of length $2l$ rocks to and fro on the top of a rough semicircular cylinder of radius a . Calculate the period of small oscillations.

13 20. A uniform solid hemisphere of radius a with its curved surface in contact with a rough horizontal table rocks through a small angle. Show that the period of small oscillations is $P = 2\pi \sqrt{\frac{11}{10g}}$. 21. The density ρ of a solid sphere of mass M and radius a varies with distance r from the centre as

Calculate the (second) moment of inertia about an axis through the centre of the sphere. Express your answer in the form of constant $\times Ma^2$. 22.

14 Two identical particles are connected by a light string of length $2ac$. The system is draped over a cylinder of radius a as shown, the coefficient of limiting static friction being μ . Determine the angle θ when the system is in limiting equilibrium and just about to slide. 23. F A mass M hangs from a light rope which passes over a rough cylinder, the coefficient of friction being μ and the angle of lap being c . What is the least value of F , the tension in the upper part of the rope, required to prevent the mass from falling? 24. x

15 A wooden cube floats on water. One of its faces is freely hinged to an axis fixed in the surface of the water. The hinge is fixed at a distance from the top of the face equal to x times the length of a side. The opposite face is submerged to a distance y times the length of a side. Find the relative density s (that is, relative to the density of the water) of the wood in terms of x and y . 25. A uniform solid sphere sits on top of a rough semicircular cylinder. It is given a small displacement so that it rolls down the side of the cylinder. Show that the sphere and cylinder part company when the line joining their centres makes an angle $53^\circ 58'$ with the vertical. 26. 12 cm 9 cm

16 A student has a triangular sandwich of sides 9 cm, 12 cm, 15 cm. She takes a semicircular bite of radius 3 cm out of the middle of the hypotenuse. Where is the centre of mass of the remainder? Is it inside or outside the bite? 27. An rubber elastic band is of length $2a$ and mass m ; the force constant of the rubber is k . The band is thrown in the air, spinning, so that it takes the form of a circle, stretched by the centrifugal force. (This takes much practice, skill and manual dexterity.) Find a relation between its radius and angular speed, in terms of a , m and k . 28. Most of us have done simple problems on friction at high school or in first year at college or university. You know the sort - a body lies on a rough horizontal table. A force is applied to it. What happens? Try this one. Find a uniform rod AB . A ruler will do as long as it is straight and not warped. Or a pencil of hexagonal (not circular) cross-section, provided that it is uniform and doesn't have an eraser at the end. Place it on a rough horizontal table. Gradually apply a horizontal force perpendicular to the rod at the end A until the rod starts to move. The end A will, of course, move forward. Look at the end B - it moves backward. There is a point C

somewhere along the rod that is stationary. I.e., the initial motion of the rod is a rotation about the point C. Calculate - and measure - the ratio AC/AB. What is the force you are exerting on A when the rod is just about to move, in terms of its weight and the coefficient of friction?

17 29. Some of the more dreaded friction problems are of the "Does it tip or does it slip?" type. This and the following four are examples of this type. A uniform solid right circular cone of height h and basal radius a is placed on an inclined plane whose inclination to the horizontal is gradually increased. The coefficient of limiting static friction is μ . Does the cone slip, or does it tip?

30. A cubical block of side $2a$ rests on a rough horizontal table, the coefficient of limiting static friction being μ . A gradually increasing horizontal force is applied as shown at a distance x above the table. Will the block slip or will it tip? Show that, if $\mu < \frac{x}{a}$, the block will slip whatever the value of x .

18 31. A cylindrical log of diameter $2a$ and mass m rests on two rough pegs (coefficient of limiting static friction μ) a distance $2ka$ apart. A gradually increasing torque C is applied as shown. Does the log slip (i.e. rotate about its axis) or does it tip (about the right hand peg)? When you've done that one, you can try a variant (which I haven't worked out and haven't posted a solution) in which a cylinder of radius a is resting against a kerb (or curb, if you prefer that spelling) of height h , and a torque is applied. Will it tip or will it slip?

19 32. This problem reveals the severe limitations of my artistic abilities, but the drawing above, believe it or not, represents a motor car seen from behind. You can see the driver and passenger. The height of the centre of mass is h and the distance between the wheels is $2d$. The car is travelling on a horizontal road surface, coefficient of friction μ , and is steering to the left in a circle of radius R , the centre of curvature being way off to the left of the drawing. As they gradually increase their speed, will the car slip to the right, or will it tip over the right hand wheel, and at what speed will this disaster take place? Fortunately, driver and passenger were both wearing their seat belts and neither of them was badly hurt, and never again did they drive too fast round a corner.

33. A uniform rod of length $2l$ rests on a table, with a length $l - a$ in contact

20 A uniform rod of length $2l$ rests on a table, with a length $l - a$ in contact

with the table, and the remainder, $l + a$ sticking over the edge - that is, a is the distance from the edge of the table to the middle of the rod. It is initially prevented from falling by a force as shown. When the force is removed, the rod turns about A. Show that the rod slips when it makes an angle θ with the horizontal, where $\tan \theta = 2 + 9(a/l)^2$. Here μ is the coefficient of limiting static friction at A. 34. A flexible chain of mass m and length l is initially at rest with one half of it resting on a smooth horizontal table, and the other half dangling over the edge. It is released, so that it starts to slide off the table. At a subsequent time t , a length $l - x$ remains in contact with the table, the remaining length $l + x$ hanging vertically, and the speed of the chain is v : $v^2 = 2gx$. Show that $v^2 = 2gx$.

21 $l(e^t - 1)X = 4e^t(e^t - 1)$ and $v = 4e^t$ 35. (a) Four books, each of width $2w$, are stacked on top of each other in a heap, thus: D What is the maximum possible overhang, D ? (b) How many books would be needed to achieve an overhang of $10w$? (c) Given an unlimited supply of books, what is the maximum overhang achievable?

APPENDIX B Solutions to Miscellaneous Problems $k = 8m$, $m = 10m$, $h = 3m$. By proportions, $h/n = h/m$ and $k/x = l/x$ and therefore $h/h = k/1$. Therefore by Pythagoras: $1 + bx^2 = 1$. Everything but x is known in this equation, which can therefore be solved for x . There are several ways of solving it; here's a suggestion. If we put in the numbers, the equation becomes $1 + 100x^2 = 1$. Put $X = 100 - x^2$, and the equation becomes

This can be written $f(X) = 3(A+B) - 1 = 0$, where A and B are obvious functions of X . Differentiation with respect to X gives $f'(X) = -(A^3 + B)$ and Newton-Raphson iteration ($X = X - f/f'$) soon gives X , from which it is then found that $x = 6.326182$ m. In the corotating frame the bob is in equilibrium under the action of three forces - its weight, the tension in the string and the centrifugal force. (If you don't like rotating reference frames and centrifugal force, it will be easy for you to do it "properly".) Resolve the forces perpendicular to the string: $m\omega^2 r \sin \theta = mg \cos \theta$, and the problem is finished. 3. (a) Raising or lowering the board doesn't apply any torques to the system, so the angular momentum L is conserved. That is, $L = m l g \sin \theta$. We also have that $g = l \cos \theta$. i. Eliminate θ from these equations. This gives: $1/\sin \theta \tan \theta - g m^2$ is constant. (2) (3) which is

ii. Eliminate I from equations (1) and (2). This gives: $2 \cot \theta = \frac{m}{l} g$, (4) L which is constant. iii. Eliminate θ from equations (1) and (2). This gives: (5) (Check the dimensions of all the equations.) Then we can get L/m from equation (1) and hence $\cos \theta = \frac{F}{l \omega^2}$: which is constant. (b) i. $\frac{1}{\sin \theta} \tan \theta = \frac{l \omega^2}{3 \sin \theta} \tan \theta = 0.023675 \text{ m}$. Although we are asked to plot θ vertically versus I horizontally, it is working out numerical values, to calculate I as a function of θ . That is, $0.287142 \text{ l} = \sin \theta \frac{3 \tan \theta}{\omega^2}$ easier, when (The number in the numerator is the cube root of 0.023675.)

θ	I
90	80
70	50
40	30
20	10

Length of string, metres

Although we are asked to plot m vertically versus I horizontally, it is easier, when working out numerical values, to calculate I as a function of ϕ . That is, $I = g \cdot a \cdot \phi$. 4 ϕ 20 18 16 14 12 10 8 6 4 2 0 ϕ 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 Length of string, metres To solve the above equation for ϕ might be slightly easier with the substitution of u for I/ϕ : $g^2 U^4 + a u^2 - 2 = 0$. With $l = 0.6$ m, this gives $u = 0.226121$ rad s, and hence $\phi = 4.422$ rad s. As in part (b) i, it is necessary to know how to solve the equation $f(x) = 0$. See section 1.4 of Celestial Mechanics if you need to know how.

6gl cos: $0(1 - \cos \theta)$ and $\therefore = (5) 3\sin\theta + 1$ Of course t) and θ) increase monotonically with θ ; but θ starts and finishes at zero, and must go through a

maximum. With $c = \cos \theta$, equation (5) can be written $\frac{1}{2} 6glc(1 - c) x - \dots$, (6) $4 - 3c^2$ and by differentiating $\frac{1}{2}$ with respect to c , we see that $\frac{1}{2}$ is greatest at an angle θ given by $3c^3 - 12c + 8 = 0$, (7) the solution of which is $\theta = 37^\circ 50'$. If the length of the rod is 1 m ($l = 0.5$ m) and $\dot{\theta} = 1$ m s⁻¹, equation (6) becomes $26.4c - 29.4c + 4 = 0$, (8) and the two solutions are $\theta = 17^\circ 15'$ and $80^\circ 52'$. The reader who has done all the problems so far will be aware of the importance of being able instantly to solve the equation $f(x) = 0$. If you have not already done so, you should write a computer or calculator program that enables you to do this instantly and at a moment's notice. See section 1.4 of Celestial Mechanics if you need to know how. If you want to find the normal reaction N of the table on the lower end of the rod, you could maybe start with the vertical equation of motion $m\ddot{y} = N - mg$. Differentiate equation (4): $\ddot{y} = \text{whatever}$, and use equation (4) again for $\dot{y}$. This looks like rather heavy and uninteresting algebra to me, so I shan't pursue it. There may be a better way...

5. In the figure below I have marked in red the forces on the rod, namely its weight mg and the horizontal and vertical components X and Y of the reaction of the hinge on the rod. I have also marked, in green, the transverse and radial components of the acceleration of the centre of mass. The transverse component is $l\ddot{\theta}$ and the radial component is the centripetal acceleration Y^2/x . From consideration of the moment of the force mg about the lower end of the rod, it is evident that the angular acceleration is $\ddot{\theta} = \frac{3g \sin \theta}{l}$, (1) 41 and by writing energy considerations, we obtain the angular speed: $\frac{1}{2} I \dot{\theta}^2 = 3gl(1 - \cos \theta)$ 21 The horizontal and vertical equations of motion are: $X = ml(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta)$ and $mg - Y = m(l\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta)$. (As ever, check the dimensions - and count the dots!) $\dot{\theta}$ as $d\theta/dt$ and integrating (with initial conditions $\theta = 0 = t = 0$), or from (2) (3) (4)

After substitution for $\dot{\theta}$ and t^2 we find $X = -mg \sin \theta (3 \cos \theta - 2)$ and $Y = \frac{1}{2} mg(1 - 3 \cos \theta)$ 2. The results follow immediately. (6) 6. Call the length of the rod $2l$. Initially the height above the table of its centre of mass is $l \cos 40^\circ$, and its gravitational potential energy is $mg l \cos 40^\circ$. When it hits the table $T(-mt \dot{y} = ml^2 \ddot{\theta})$. Therefore, at angular speed $\dot{\theta}$, its kinetic energy is $\frac{1}{2} 3g \cos 40^\circ$ 21 - $4.746 \text{ rad s}^{-1} = 271.9 \text{ degs}$. To find the time taken, you can use equation 9.2.10: $\frac{1}{2} I \dot{\theta}^2 = mgl(1 - \cos \theta)$ To 400 4E-r(o) (7) Here, $I = ml^2$ $E = mgl \cos 40^\circ$ $V(\theta) = mgl \cos \theta$ and therefore $40 \cos 40^\circ = \cos \theta$ (8) The

magnitude of the quantity before the integral sign is 0.184428 s. To find the value of the integral requires either that you be an expert in elliptic integrals or (more likely and more useful) that you know how to integrate numerically (see Celestial Mechanics 1.2.) I make the value of the integral 2.187314, so that the time taken is 0.4034 seconds. When integrating, note that the value of the integrand is infinite at the lower limit. How to deal with this difficulty is dealt with in Celestial Mechanics 1.2. It cannot be glossed over.

7. Here is the diagram. The forces are the weight mg of the rod, and the force of the table on the rod. However, I have resolved the latter into two components - the normal reaction N of the table on the rod, and the frictional force F , which may be either to the left or the right, depending on whether rod is tending to slip towards the right or the left. The magnitude of F is less than gN as long as the rod is not just about to slip. When the rod is just about to slip, $F = gN$, g being the coefficient of limiting static friction. N mg F Just as in Problem 5, the equations of motion, as long as the rod does not slip, are $F = -mg \sin \theta (3 \cos \theta - 2)$ (1) and $N = \frac{1}{2} mg (1 - 3 \cos \theta)$: (2) $F = 3 \sin \theta (3 \cos \theta - 2)$ (3) $N = (1 - 3 \cos \theta)$: The figure below shows F/N as a function of θ . One sees that, as the rod falls over, F/N increases, and, as soon as it attains a value of g , the rod will slip. We see, however, that F/N reaches a maximum value, and by calculus we can determine that it reaches a maximum value of $15 \sqrt{3}/128 = 0.3706$ when $\theta = \cos^{-1}(1/3) = 35^\circ 05'$. If $g < 0.3706$, the bottom of the rod will slip before $\theta = 35^\circ 05'$. If, however, $g > 0.3706$, the rod will not have slipped by the time $\theta = 35^\circ 05'$, and it is safe for a while as F/N starts to decrease. When θ reaches $\cos^{-1}(2/3) = 48^\circ 11'$, the frictional force changes sign and thereafter acts to the left. (The frictional force of the table on the rod acts to the left; the frictional force of the rod on the table acts to the right.) We know by now (since the rod survived slipping before $\theta = 35^\circ 05'$, that the magnitude of F/N can be at least as large

0.2 -0.2 -0.6 -1 0 -0.8 5 10 15 20 25 30 35 40 45 50 55 1 θ , degrees as 0.3706, and it doesn't reach this until $\theta = 51^\circ 15'$. Therefore, if the rod hasn't slipped by $\theta = 35^\circ 05'$, it won't slip before $\theta = 51^\circ 15'$. But after that it is in danger again of slipping. F/N becomes infinite ($N = 0$) when $\theta = \cos^{-1}(2/3) = 70^\circ 31'$, so it will certainly slip (to the right) before then. If $p = 0.25$, the rod will slip to the left when $3 \sin \theta (3 \cos \theta - 2) = 1$ ($1 - 3 \cos \theta$): $-\cos \theta = 4$ or $\theta = 19^\circ 39'$. If $p = 0.75$, the rod will slip to the right when $3 \sin \theta (3 \cos \theta - 2) = 1$

2) $3 - \cos\theta = 5/3$ (1-3 cos θ): 4' Again, it is very necessary that you prepare for yourself a program that will instantly solve the equation $f(x) = 0$.

o P o Let the length of the ladder be $2l$. By geometry, the distance OC remains equal to l throughout the motion; therefore C describes a circle of radius l , centre O. I have marked in, in green, the radial and transverse components of the acceleration of C, namely $l\dot{\theta}^2$ and $l\ddot{\theta}$. The angular speed of the ladder is $\dot{\theta}$ and the linear speed of the centre of mass C is $l\dot{\theta}$. I have also marked, in red the three forces acting on the ladder, namely its weight and the reactions of the floor and the wall on the ladder. The angular speed $\dot{\theta}$ can be obtained from energy considerations. That is, the loss of potential energy in going from angle θ to the vertical to angle 0 is equal to the gain in translational and kinetic energies: $2mgl(\cos\theta - \cos 0) = \frac{1}{2}m(l\dot{\theta})^2 + \frac{1}{2}I\dot{\theta}^2$. (1) 21 The angular acceleration $\ddot{\theta}$ can be obtained by considering that the total moment of all forces about P is $mg l \sin \theta$, and the rotational inertia is $\frac{1}{3}ml^2$: $mg l \sin \theta = \frac{1}{3}ml^2 \ddot{\theta}$.

The vertical and horizontal equations of motion are: $N \cos \theta = m(l\dot{\theta}^2 \cos \theta - g \sin \theta)$ (3) and $mg - N = m(l\dot{\theta}^2 \sin \theta + l\ddot{\theta} \cos \theta)$, (4) although we need only the first of these, because we wish to find out when $N = 0$. On substitution for $\dot{\theta}^2$ and $\ddot{\theta}$ we find that $3 \cos \theta - 2 \cos \theta = 5 \cos \theta$ (5) $N = 7$ and $N = mg(1 - 6\cos\theta\cos\theta + 9\cos^2\theta)$. (6) We need only the first of these to see that N becomes zero (and hence the upper end loses contact with the wall) $\cos \theta = 5/3$.

It will, I think, be agreed that the point O remains fixed in space as long as the semicylinder remains in contact with wall and floor. Therefore the centre of mass C moves in a circle around O. We'll call the radius of the circle, which is the distance between O and C, b , which, for a semicylinder, equals $4a/(3\pi)$ (see Chapter 1), where a is the radius of the semicylinder. I have marked, in red, the three forces on the semicylinder, and also, in green, the radial and transverse components of the acceleration. The angular speed $\dot{\theta}$ can be obtained from energy considerations. The gain in kinetic energy in going from rest to an angular speed $\dot{\theta}$ is $\frac{1}{2}mk^2\dot{\theta}^2$, and the gain in potential energy when the centre of mass drops through a vertical distance $b \sin \theta$ is $mbg \sin \theta$. Here k is the radius of gyration about O, which, for a semicylinder, is given by $k^2 = \frac{1}{2}b^2 + a^2$ [I have left b and k as they are in the equations, so that

the analysis could easily be adapted, if needed, for a hollow semicylinder, or a solid hemisphere, or a hollow hemisphere. From Chapters 1 and 2 we recall: Solid semicylinder: $4a^2 b^2 \sin^2 \theta = k^2 \omega^2$ [2] Hollow semicylinder: $2a^2 b^2 \sin^2 \theta = k^2 \omega^2$ [2] Solid hemisphere: $b^2 \sin^2 \theta = 8k^2 \omega^2$ [2] Hollow hemisphere: $b^2 \sin^2 \theta = k^2 \omega^2$ [2] On equating the gain in kinetic energy to the loss in potential energy, we obtain $2b \sin \theta = k^2 \omega^2$. The angular acceleration α can be obtained from applying $\tau = I\alpha$ about O: $mgbc \cos \theta = mk^2 \alpha$, (1) from which $b \sin \theta = r \cos \theta$. (2) The horizontal and vertical equations of motion are

$$N^2 = mb(t^2 \cos \theta + \frac{1}{2} \sin \theta) \quad (3) \text{ and } N - mg = mb(t^2 \sin \theta - \frac{1}{2} \cos \theta). \quad (4)$$

We don't really need equation (4), because we are trying to determine when $N = 0$. On substitution from equations (1) and (2), equation (3) becomes $6mb^2 g \sin \theta - 2 \sin \theta \cos \theta$. (5) This is zero when $\theta = 0$ (which was the initial condition) or when $\theta = 90^\circ$, at which point contact with the wall is lost, which it was required to show. At this instant, the rotational velocity is $\omega = \sqrt{2g/b}$, and the linear velocity of C is $\sqrt{2g}$ horizontally to the right. The rotational kinetic energy is $\frac{1}{2} I \omega^2$, where $I = \frac{1}{2} k^2$, and I is the rotational inertia about the centre of mass, which is $m(k^2 - b^2)$. $mbg(k^2 - b^2)$ The translational kinetic energy is $\frac{1}{2} mv^2$, where $v = \sqrt{2g}$. $\frac{1}{2} mb(3g)$ The sum of these is mbg , which is just equal to the loss of the original potential energy, which serves as a check on the correctness of our algebra. There are now no horizontal forces, so the horizontal component of the velocity of C remains constant. The semicylinder continues to rotate, however, until the rotational kinetic energy is converted to potential energy and C rises to its maximum height. If the base then makes an angle θ with the vertical, the gain in potential energy is $mbg \sin \theta$, and equating this to the rotational kinetic energy gives

$\sin \theta = \frac{1}{3}$ This gives the following results: Solid semicylinder: $\theta = 30^\circ$ Hollow semicylinder: $\theta = 46^\circ 36'$ Solid hemisphere: $\theta = 39^\circ 46'$ Hollow hemisphere: $\theta = 25^\circ 38'$ It is well known that if $\theta > \tan^{-1} g$ the particle will slide down the plane unless helped by an extra force. I have drawn the three forces acting on the particle. Its weight mg . The reaction R of the plane on the particle; if the particle is in limiting static equilibrium, this reaction will make an angle X ("the angle of friction") with the plane such that $\tan X = g$. It therefore makes an angle $\theta - X$ with the vertical. Finally, the

additional force P needed; we do not initially know the direction of this force. When three (or more) coplanar forces are in equilibrium and are drawn head-to-tail, they form a closed triangle (polygon). I draw the triangle of forces below.

R mg It will be clear from the triangle that P is least when the angle between P and R is 90° : P R mg The least value of P is therefore $mg(\sin c \cos X - \cos c \sin X)$. But $\tan = \text{it}$ and therefore $\sin - \text{it}$ and $\cos - 1$ $mg(\sin c - \text{it} \cos c)$

11. As the cylinder rolls down the plane, the wedge, because its base is smooth, will slide towards the left. Since there are no external horizontal forces on the system, the centre of mass of the system will not move horizontally (or, rather, it won't accelerate horizontally.) As usual, we draw a large diagram, using a ruler, and we mark in the forces in red and the accelerations in green, after which we'll apply $F = ma$ to the cylinder, or to the wedge, or to the system as a whole, in two directions. It should be easy and straightforward. I have drawn the linear acceleration g of the cylinder down the slope, and its angular acceleration $\{\}$. I have drawn the linear acceleration 5 of the wedge, which is also shared with the cylinder. I have drawn the gravitational force mg on the cylinder. There is one more force on the cylinder, namely the reaction of the wedge on the cylinder. But I'm not sure in which direction to draw it. Is it normal to the plane? That would mean there is no frictional force between the cylinder and the plane. Is that correct (remembering that both the cylinder and the wedge are accelerating? Of course I could calculate the moment of the force mg about the point of contact of the cylinder with the plane, and then I wouldn't need to concern myself with any forces at that point of contact. But then that point of contact is not fixed. Oh, dear, I'm getting rather muddled and unsure of myself. This problem, in fact, is ideally suited to a lagrangian rather than a newtonian treatment, and that is what we shall do. Lagrange proudly asserted that it was not necessary to draw any diagrams in mechanics, because it could all be done analytically. We are not quite so talented a Lagrange, however, so we still need a large diagram drawn with a ruler.

But, instead of marking in the forces and accelerations in red and green, we mark in the velocities in blue. No frictional or other nonconservative forces do any work, so we can use Lagrange's equations of motion for a

conservative holonomic system; ($\dot{\theta} = \dot{\phi}$). The speed of the wedge is $\dot{x}$ and the speed of the centre of mass of the cylinder is $\sqrt{\dot{x}^2 + 2\dot{x}\dot{\theta}a\cos\theta + a^2\dot{\theta}^2}$, and the angular speed of the cylinder is $\dot{\theta}$. The kinetic energy of the system is $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{\theta}^2 + m\dot{\theta}^2 a^2 \cos^2\theta$, or $T = \frac{1}{2}(m + M)\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2$, and the potential energy is $V = mgs\sin\theta$. Application of Lagrange's equation to the coordinate x gives us

$\cos\theta : (m + M)\ddot{\theta} + m\ddot{x} = g\sin\theta$. Elimination of $\ddot{x}$ from these two equations gives us $mg\sin\theta\cos\theta = (m + M)\ddot{\theta}$. You can also easily find an expression for g if you wish. If the wedge had been fixed and immovable, and not free to slide on a smooth table, you would easily have been able to obtain g by newtonian methods. If you put $M = \infty$ in the expression for g for the smooth table, do you get the same as you do for the immovable wedge? (You should do.) Now that you have found $\ddot{\theta}$, you can also find $\dot{\theta}$. You therefore now know the magnitude and direction of the velocity of the centre of mass of the cylinder. From this you should be able to find the net force and torque on the cylinder, and from these, you should be able to find the magnitude and direction of the reaction of the wedge on the cylinder. I have not pursued this. If anyone succeeds in doing so, and would like his/her solution posted here, with acknowledgment, let me know.

12 x y N There is no acceleration normal to the plane, and therefore $N = mg\cos\theta$. The frictional force F acts along the tangent to the path and is equal to μN , or $\mu mg\cos\theta$, where μ is the

coefficient of moving friction. We are told to ignore the difference between the coefficients of moving and limiting static friction. Since the particle was originally at rest in limiting static friction, we must have $\mu = \tan\theta$. Therefore $F = mg\sin\theta$. The tangential equation of motion is $m\ddot{s} = -F + \text{whatever the component of } mg \text{ is in the tangential direction in the sloping plane}$. The component of mg down the plane would be (look at the left hand drawing) $mg\sin\theta$, and so its tangential component (look at the right hand drawing) is $mg\sin\theta\sin\theta$. So we have, for the tangential equation of motion, $m\ddot{s} = -mg\sin\theta + mg\sin\theta\sin\theta$, or $\ddot{s} = -g\sin\theta(1 - \sin\theta)$. We are seeking a relation between V and s , so, in the now familiar fashion, we write $V dV$ for $\ddot{s} ds$, so the tangential equation of motion is $\frac{1}{2}dV^2 = -gs\sin\theta(1 - \sin\theta)$. (1) ds We also

need the equation of motion normal to the trajectory. The component of mg in that direction is $mg \sin \theta \cos \phi$, and so the normal equation of motion is $mV^2/p - mg \sin \theta \cos \phi$. Here p is the radius of curvature of the path, which is the reciprocal of the curvature $ds/d\theta$. The normal equation of motion is therefore $V^2/d = g \sin \theta \cos \phi$. (2) ds Divide equation (1) by equation (2) to eliminate s and thus get a desired differential equation between V and θ : $1/V dV = (1 - \sin^2 \theta) d\theta$. (3) $V d\theta = \cos \theta$ This is easily integrated; a convenient (not the only) way is to multiply top and bottom by $1 + \sin \theta$. In any case we soon arrive at

$\ln V = -\ln(1 + \sin \theta) + \text{constant}$, (4) and with the initial condition $V = V_0$ when $\theta = 0$, this becomes $V = V_0 / (1 + \sin \theta)$. In the limit, as $\theta \rightarrow 90^\circ$, $V \rightarrow V_0/2$. The particle is then moving at constant velocity and is in equilibrium under the forces acting upon it just when it was initially at rest. 13. M = mass of complete sphere of radius a . M_2 = mass of missing inner sphere of radius xa . M = mass of given hollow sphere. We have $M = M_2 - M$ and $M_2/M = x^3$, and therefore $M = Mx^3/(1-x^3)$ and $M_2 = Mx^3/(1-x^3)$. Also $I = \frac{2}{5} M a^2$. $I = \frac{2}{5} M a^2 x^5 / (1-x^3)^2$. Hence $I = \frac{2}{5} M a^2 x^5 / (1-x^3)^2$. If $x = 0$, $I = 0$, as expected. If $x = 1$, you may have to use de l'Hôpital's rule to show that $I \rightarrow \frac{2}{5} M a^2$, as expected. 14. M = mass of mantle. M_2 = mass of core. $M = M_2 + M$ and $M_2/M = (1-x)^3$, and therefore $x^3 = M_2/M$ and $S(1-x^3) = M_2/M$. $M_2 = M x^3 / (1-x^3)$. $S(1-x^3) = M_2/M$.

Also $I = \frac{2}{5} M a^2 x^5 / (1-x^3)^2$. $I = \frac{2}{5} M a^2 x^5 / (1-x^3)^2$. where I have made use of the result from the previous problem. expressions for M and M_2 , we quickly obtain On substitution of the $s = (1-x)^3$ $I = \frac{2}{5} M a^2 x^5 / (1-x^3)^2$. A hollow planet would correspond to $1/s = 0$. Divide top and bottom by s and it is immediately seen that the expression for a hollow planet would be identical to the expression obtained for the previous problem.

0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Core size																		

 Note that both $x = 0$ and $x = 1$ correspond to a uniform sphere, so that in either case, $I = \frac{2}{5} M a^2$; for all other cases, the moment of inertia is less than $\frac{2}{5} M a^2$. The core size for minimum moment of inertia is easily found by differentiation of the above expression for I , and the required expression follows after some algebra. For $s = 0.6$, the equation becomes $9 - 15x^2 - 4x^5 = 0$, of which the only positive real root is $x = 0.736382$, which corresponds to a moment of inertia of $0.90376 \times \frac{2}{5} M a^2$.

Note 2.2 that, for $s = 0.6$, the moment of inertia, expressed in units of $\frac{1}{2}Ma^2$, varies very little as

the core size goes from 0 to 1, so that measurement of the moment of inertia places very little restriction on the possible core size. The inverse of equation (1) is $(1-x)x^5 - I(1-x)x^3 + (1-I)x : 0$, (2) x^2 where I is expressed in units of $\frac{1}{2}Ma^2$. For $I = 0.911$, there are two positive real roots (look at the graph); they are $x = 0.64753$ and 0.81523 . For $I = 0.929$, the roots are 0.55589 and 0.87863 . Thus the core size could be anything between 0.55589 and 0.64753 or between 0.81523 and 0.87863 a rather large range of uncertainty. Even if I were known exactly (which does not happen in science), there would be two solutions for X . 15. This is just a matter of geometry. If, when you make a small angular displacement, you raise the centre of mass of the brick the equilibrium is stable. For, while the brick is in its vertical position, it is evidently at a potential minimum, and you have to do work to raise the centre of mass. If, on the other hand, your action in making a small angular displacement results in a lowering of the centre of mass, the equilibrium is unstable. When the brick is in its vertical position, the height h_0 of its centre of mass above the base of the semicylinder is just $h_0 = R + I$. When it is displaced from the vertical by an angle θ , the point of contact between brick and semicylinder is displaced by a distance $R\theta$, and, by inspection of the drawing, the new height h is $h = R\cos\theta + R\theta\sin\theta + \frac{1}{2}R\theta^2/\cos\theta$. $\therefore h - h_0 = R\theta\sin\theta - (R + I)(1 - \cos\theta)$. If you Maclaurin expand this as far as θ^2 , you arrive at $h - h_0 = (I - R)\theta^2$. This is positive, and therefore the equilibrium is stable, if $I < R$, or $2I < 2R$, i.e. if the length of the brick is less than the diameter of the semicylinder.

16. As in the previous question, it is just a matter of geometry. If rolling the Thing results in raising its centre of mass, the equilibrium is stable. Initially, the height of the centre of mass is $h_0 = b + I$. After rolling, the dashed line, which joins the centres and is of length $a + b$, makes an angle θ with the vertical. The short line joining the centre of mass of the Thing to the centre of curvature of its bottom is of length $l - a$ and it makes an angle $\theta + \phi$ with the vertical. The height of the centre of mass is therefore now $h = (a+b)\cos\theta + (l-a)\cos(\theta+\phi)$. The centre of mass has therefore rise through a height $h - h_0 = (a+b)\cos\theta + (l-a)\cos(\theta+\phi) - b - I$. Also, the two angles are related by $a\theta = b\phi$, so that

$h - h_0 = (a+b)\cos\theta + (-a)\cos[1 + (b/a)\theta] - b - I$. Maclaurin expand the cosines to θ^2 and you should get $h - h_0 = -\frac{3}{2}a^2[b + (-a)(1+b/a)]$. For stability this must be positive, and hence $- > -- + -$ $\frac{I}{a} \frac{b}{1} \frac{1}{Ira} = b$, this becomes $l < \frac{I}{a}$. For a hollow semicylinder, $I = (1 - 2/x)a^2 = 0.363a^2$. For a hollow hemisphere, For a solid semicylinder, For a solid hemisphere, $\therefore$ Stable $I = 0.5a^2$. $\therefore$ Borderline stable $I = [1 - 4/(3r)]a^2 = 0.576a^2$. $\therefore$ Unstable $I = 5.0625a^2$. $\therefore$ Unstable $\frac{I}{a} = 17$. We need to find the height h of the centre of mass above the level of the pegs as a function of θ . See drawing on next page. Angles: $BAC = 45^\circ - \theta$ $ABX = 45^\circ + \theta$ Distances: $AB = 2ac$ $AC = 2ac \cos(45^\circ - \theta)$ $EF = 2ac \cos(45^\circ - \theta) \cos(45^\circ + \theta) = ac \cos 2\theta$ $DC = DF$: $ax/\cos\theta$ $h = DF - EF = a(x/\cos\theta - c \cos 2\theta)$ $h_0 =$ height of centre of mass above pegs when $\theta = 0$ $\frac{dh}{d\theta} = a(x/\sin\theta - 2c \sin 2\theta)$ $y = h_0 - c \frac{dy}{d\theta}$ will show that maxima and minima of y (and of the potential energy), and hence equilibria, occur for $\theta = 0$ and for $\cos\theta = 1/(cx/a)$, which is possible only if $c > 1/x$.

A second differentiation will show which extrema are maxima and which are minima. In particular the second derivative at $\theta = 0$ is zero for $c = 1/x$. I draw graphs of y for several c below. If $c = -$, $\theta = 45^\circ$.

2.5 0.75 0.5 0.5 -80 -60 -40 -20 0 , degrees 20 40 60 80

There are three forces acting on the hemisphere: Its weight mg . The reaction N of the wall, which is perpendicular to the wall since the wall is smooth. The reaction R of the floor, which acts at an angle L to the floor, where $\tan L = \mu$. Three forces in equilibrium must act through a point; therefore all three forces act through the point P . It is thus clear that $OP \sin \theta = OC$ $\frac{I}{a} = 0 = 41^\circ 48'$. If $\mu = 3$, $\theta = 90^\circ$. If $\mu > 3$, the hemisphere can rest in any position, the equilibrium not being limiting static equilibrium. 19. $c = a/O$ P When the rod makes an angle θ to the horizontal, the distance from its centre of mass C to the point of contact P is a/O . The moment of inertia about P is I_P torque about P is $mg a/O \cos \theta$. The equation of motion is $m(l + a/O) \ddot{\theta} = -mg a/O \cos \theta$. To first order in θ , $\cos \theta = 1$ and $a/O \ll l$. Therefore, to first order, the equation of motion is $\ddot{\theta} = -ga/O$

2rl and so the period is $P = 2\pi \sqrt{O/g}$. I have drawn only one force. I am going to consider the rotational equation of motion about A , so I shan't be

concerned with the force at A. The equation of motion about A is $I \ddot{\theta} = -mgx$, where I is the moment of inertia about A. The moments of inertia about O, C and A, respectively: I_O , $I_C = I_O - m a^2$, $I_A = I_O - m a^2 + m b^2$, where I have made good use of the parallel axes theorem. The distance b is given by the cosine rule to triangle OCA, and hence we find, after a little algebra, for the moment of inertia about A: $I_A = I_O - m a^2 + m b^2 \cos \theta$. 5 We also need

$3a \sin \theta$ and hence we obtain for the equation of motion $\ddot{\theta} = -15g \sin \theta / a(56 - 30 \cos \theta)$. For small angles, this becomes $\ddot{\theta} = -15g \theta / a$. From the sine rule we have $26a \sin \theta = -15g \theta$. $\sin 4 \sin(\theta + 3a/a)$ and replacing the sines by the angles, for small angles, we find that $\ddot{\theta} = -25g \theta / a$. equation of motion becomes $\ddot{\theta} = -25g \theta / a$ and so the period is $P = 2\pi \sqrt{a/25g}$. Thus the (second) moment of inertia with respect to the centre (see chapter 2.19 of chapter 2) is $I_{\text{centre}} = \int_0^a 4\pi r^2 (r^2/2) dr = 2\pi r^4/5$. The moment of inertia with respect to an axis through the centre is $2/3$ of this: $I_{\text{axis}} = \frac{2}{3} I_{\text{centre}} = \frac{2}{3} \pi r^4/5 = \frac{2}{15} \pi r^4$.

22. $N_1 = mg \cos(\theta - \theta)$ $F_1 = IN_1 \sin \theta + F_2 mg \cos(\theta + \theta)$ Left-hand particle: $T : mg [\sin \cos(\theta - \theta) + \sin(\theta - \theta)]$. Right-hand particle: $T : mg [\sin(\theta + \theta) - \sin \cos(\theta + \theta)]$. $\therefore \sin [\cos(\theta - \theta) + \cos(\theta + \theta)] = \sin(\theta + \theta) - \sin(\theta - \theta)$, and, by the "sum and difference" trigonometrical formulae, we obtain $2 \cos \theta \cos \theta = 2 \cos \theta \sin \theta$, from which $\tan \theta = 1/3$. $T = \frac{1}{3} mg$ $N_1 = T$

Consider a portion of the rope between 0 and 80 . There are four forces on this portion. The tension T at 0. The tension $T + 8T$ at $0 + 80$ ($8T$ is negative). The normal reaction $8N$ of the cylinder on the rope. The frictional force $[8T \sin \theta]$ of the cylinder on the rope. Note that the rope is about to slip downwards, so the friction force is upwards as shown. We have $8N = (2T + 8T) \sin(\theta)$ $(T + 8T) \cos(\theta) + [8T \sin \theta] = T \cos(\theta)$. To first order, these become $8N = T \sin \theta$ and $8T = -T \sin \theta$. $\therefore 8T = -T \sin \theta$ and hence by integration $F = Mg e^{-\sin \theta}$. 24 $2ax$ $2ay$ X' -axis Y' -axis

Area of square = $4a^2$ Area of rectangle = $4a^2 (1 - x)$ Area of triangle = $2a^2(x + y - 1)$ Area of trapezoid = $2a^2(1 - x + y)$ The weight of the cube is $8a^3 \rho_s g$, and it acts downward through C, the centre of mass. The hydrostatic upthrust is $4a^3(1 - x + y) \rho_f g$ and it acts upward through the centre of buoyancy H. Here ρ_f is the density of the fluid, and ρ_s is the density of the wood. We evidently must find the X' -coordinate of C and of H. Let's first of all find the X' - and Y' -coordinates (see the next figure). The X' - and Y' -coordinates of C are

trivial and quite easy respectively: $X_c = a$ $Y_c = a(1-2x)$ You are going to have to work quite hard at it to find the X- and Y- coordinates of H, the centre of buoyancy, which is the centroid of the trapezoid. "After some algebra" you should find $2(1-x+2y)a$ $2(2-4x+2y+2x-2xy-y)a$ $3(1-x+y)$ $3(1-x+y)$ X-axis Y-axis

To find the X'-coordinates of C and of H, we use the usual formulas for rotation of axes, being sure to get it the right way round: $Y' \sin \theta \cos \theta$ ' together with $\tan \theta = x + y - 1$. Take moments about the axle (origin): $8a^3 \rho g X'_c : 4a^3 \rho g(1-x+y)$. After a little more algebra, you should eventually arrive at $3(2-3x-y+2x^2+2xy)$ $3-7x+2y+6x^2-3y^2-2x^3+y^3+3xy : 25$.

Let the radii of the cylinder and sphere be a and b respectively, and the mass of the sphere be M . The angles θ and ϕ are related by $a \sin \theta = b \sin \phi$. I have drawn the three forces on the sphere, namely its weight, the normal reaction of the cylinder on the sphere, and the frictional force on the sphere. The transverse acceleration of the centre of the sphere is $(a+b)\ddot{\theta}$ and the centripetal acceleration is $(a+b)\dot{\theta}^2$. The equations of motion are: $Mg \sin \theta - F = M(a+b)\ddot{\theta}$ and $Mg \cos \theta - N = M(a+b)\dot{\theta}^2$. (2) (1) The angular acceleration of the sphere about its centre is $\ddot{\phi} = (1+a/b)\ddot{\theta}$, and its rotational inertia is $\frac{2}{5}Mb^2$. The torque that is causing this angular acceleration is Fb , and therefore the rotational equation of motion is $Fb = \frac{3}{5}Mb^2(1+a/b)\ddot{\theta}$. (3) Elimination of F between equations (1) and (3) yields $\ddot{\theta} = \frac{5g \sin \theta}{7(a+b)}$. (4) $\frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta} \frac{d\theta}{dt}$ in the usual way and integrate with initial conditions $\theta = 0 = t = 0$, or from energy considerations: $\frac{1}{2}M(a+b)\dot{\theta}^2 = [0g(1-\cos\theta)] \cdot \frac{7}{2}(a+b)$ Substitute for $\dot{\theta}$ and $\ddot{\theta}$ into equation (2) to obtain $N = Mg(17\cos\theta - 10)$. (6) This is zero, and the sphere leaves the cylinder, when $\cos \theta = 10/17$, $\theta = 53^\circ 58'$.

26. 12 cm 9 cm 2 Surface density = c g cm Original sandwich: Mass = $54c$ g x-coordinate of centre of mass = 3 cm y-coordinate of centre of mass = 4 cm Bite: Mass = $c \cdot 32(5) = 14.13716694c$ g Distance of centre of mass from hypotenuse = $\frac{4}{3} = 1.273239545$ cm 4. 16 x-coordinate of centre of mass = $4.5 - \frac{4}{3} \sin \theta = 4.5 - 3.481408364 \cos \theta$

y-coordinate of centre of mass = $6 - 4 \cos \theta = 6 - 5.236056273 \cos \theta$ Remainder: Mass = $(54 - 14.13716694)c = 39.86283306c$ g x-coordinate of centre of mass y-coordinate of centre of mass Moments: $39.86283306 + 14.13716694x \cdot 3.481408364 = 54x \cdot 3.481408364 = 2.829270780cm$ $39.86283306y +$

14.137 166 94x5.236056273 = 53x4. $y = 3.561638436$ cm This point is very close to the edge of the bite. The centre of the bite is at (4.5, 6), and its radius is 3. Its equation is therefore $(x-4.5)^2 + (y-6)^2 = 9$, or $x^2 + y^2 - 9x - 12y + 47.25 = 0$. The line $x = 2.829270780$ cuts the circle where $y^2 - 12y + 29.79133613 = 0$. The lower of the two points of intersection is at $y = 3.508280941$ cm. The centre of mass is slightly higher than this and is therefore just inside the bite. 27. T

Consider a portion of the band within the angle 80° . Its mass is m . When the band is spinning at angular speed ω and its radius is r , the centrifugal force on that portion is $m\omega^2 r$ - (I leave it to the philosophers and the schoolteachers to debate as to whether there "really" is "such thing" as centrifugal force - I want to get this problem done, and I'm referring to a co-rotating frame.) The y-component of this force is $m\omega^2 r \cos 40^\circ$. Also, the tension in the band when its radius is r is $T = 2kr(r - a)$. Consider the equilibrium of half of the band. The y-component of the centrifugal force on it is $m\omega^2 r \cos 40^\circ$. The opposing force is $2T = 4kr(r - a)$. Equating $m\omega^2 r \cos 40^\circ = 4kr(r - a)$ these gives $m = 4k(r - a) / \omega^2 \cos 40^\circ$. 28. Let the distance AB be l and the distance AC be c . Let the mass of the rod be m . A P C B F x Consider an elemental portion dx of the rod at P at a distance x from A. Its weight is $m g dx$. When the rod is about to move, it will experience a frictional force f [tmg dx which will be in the direction shown if P is to the left of C, and in the opposite direction if P is to the right of C. When the rod is just about to move (but has not yet done so) it is still in equilibrium. Consider the moment about A of the frictional forces on the rod. The clockwise moment of the frictional forces on AC must equal the counterclockwise moment of the frictional forces on CB. Thus

The net force on the rod is $F \sin \theta$ and this is zero, and therefore $\tan \theta = 2c / l$. 29. The cone slips when $\tan \theta > 4a / h$. It tips when C (the centre of mass) is to the left of M. The distance OC is $h/4$. (See Chapter 1, section 1.7). Therefore it tips when $\tan \theta > 4a / h$. Thus it slips if $\tan \theta < 4a / h$ and it tips if $\tan \theta > 4a / h$.

30. 26/ x A When the block is just about to tip, the reaction of the table on the block acts at A and it is directed towards the point K, because, when three coplanar forces are in equilibrium they must act through a single point. The

angle L is given by $\tan L = a/x$. However, by the usual laws of friction, the block will slip as soon as $\tan L = \mu$. Thus the block will slip if $\mu < a/x$, and it will tip if $\mu > a/x$. Expressed otherwise, it will slip if $x < a/\mu$ and it will tip if $x > a/\mu$. The greatest possible value of x is $2a$; therefore the block will inevitably slip if $\mu < 1/2$. 31. When or if the cylinder is just about to tip, it is about to lose contact with the left hand peg. The only forces on the cylinder are the torque, the weight, and the reaction R of the right hand peg on the cylinder, which must be vertical and equal to mg . But the greatest possible angle that the reaction R can make with the surface of the cylinder is the angle of

friction L given by $\tan L = \mu$. From geometry, we see that $\sin \theta = k$, or $\tan \theta = k/x\sqrt{1-k^2}$. Thus the cylinder will slip before it tips if $\mu < k/x\sqrt{1-k^2}$ and it will tip before it slips if $\mu > k/x\sqrt{1-k^2}$. If the cylinder tips (which it will do if $\mu > k/x\sqrt{1-k^2}$), the clockwise torque z at that moment will equal the counterclockwise torque of the couple (R and mg), which is $mgka$. Thus the torque when the cylinder tips is $TIP: c : mgka$. (1) pJV2 When or if the cylinder is just about to slip, the forces are as shown above, in which I have resolved the reactions of the pegs on the cylinder into a normal reaction (towards the axis of the cylinder) and a frictional force, which, when slipping is about to occur, is equal to it times the normal reaction. The equilibrium conditions are $p(N + N)\cos\theta + (N - N)\sin\theta = 0$, $[t(N - N)\sin\theta - (N + N)\cos\theta + mg] = 0$ and $p(N + N^2)a = c$. We can find $N + N^2$ by eliminating $N - N^2$ from the first two equations, and then, writing $x/\sqrt{1-k^2}$ for $\cos \theta$, we find that, when slipping is about to occur,

SLIP $T = mga$ x I have drawn below the functions $p\sqrt{1-k^2}$ (tip) and $--- x$ (slip) $mg a \sqrt{1 + p^2 x^2}$ for $k = 0.1, 0.3, 0.5, 1/x$ and 0.9 . The horizontal lines are the tip functions, and the curves are the slip functions. As long as $p < k/x\sqrt{1-k^2}$: the cylinder will slip. As soon as $p > k/x\sqrt{1-k^2}$: the cylinder will tip. 1.2 0.8 0.4 0.2 0 0.5 1 1.5 2 2.5 0.9 1/x/2 0.5 0.3 0.1

32. $F_1 = mg$ $F_2 = 2d/r$, mv^2/l We'll leave to the philosophers the question as to whether centrifugal force "really exists", and we'll work in a co-rotating reference frame, so that the car, when referred to that frame, is in static equilibrium under the six forces shown. Clearly, N and $N^2 = mg$ and $F + F_2 = m v^2/R$. The car slips when $F + F_2 = g(N + N^2)$; that is, when $v = x/ggR$. The

car tips when $mt;2h/R = mgd$; that is, when $t; = x/-R/h$. That is, it will slip or tip according as to whether $g < d/h$ or $> d/h$. For example suppose $d = 60$ cm, $h = 80$ cm, $g = 9.8$ m s⁻², $R = 30$ m, $g = 0.8$. In that case, $d/h = 0.75$, so it will tip at $t; = 14.8$ m s⁻¹ = 53.5 km hr⁻¹. But if it rains, reducing g to 0.7, it will slip at $t; = 14.3$ m s⁻¹ = 51.6 kmhr⁻¹.

33. N a} F mg I have drawn in green the radial and transverse components of the acceleration of the centre of mass at) 2 and ai respectively. I have drawn in red the weight of the rod and the normal and frictional components of the force of the table on the rod at A, N and F respectively. The following are the equations of motion: Normal: $mai = mg\cos\theta - N$. (1) Lengthwise: $mat) = -mg \sin \theta + F$. (2) Rotation: $k:i = ga \cos \theta$. (3) Here k is the radius of gyration about A, given by $k^2 = -l^2 + a^2$. (4) From equations (1), (3) and (4), we obtain $N = mg\cos\theta..12 + 3a^2$. (5) The space integral (see Chapter 6, section 6.2) of equation (3), with initial condition $t) = 0$ when $\theta = 0$, results in

)2 _ ga sinO. 2k 2 Combining this with equations (2) and (4) leads to At the instant of slipping, $F = \tan \theta - mg \sin \theta$. (2) $12 + 9a^2 (2/2 + 6a^2 gN$, and hence, from equations (5) and (7) we find $2p^2 + 9(a/l)^2$ (6) (7) 90 80 [0 70 60 50 40 30 20 10 p=2 0 I I I I I I I I 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 a/I

34. I derive $v^2 = gx + g--x$ by two different methods - one from 1 considerations, the other from angular momentum considerations. First, energy. energy If the table top is taken to be the zero level for potential energy, the initial potential - -mgl. energy was $\frac{1}{2}m.g.71 =$ When the length of the dangling portion is $l + x$, the potential energy is $(-)\frac{1}{2}l + xl g \frac{1}{2}(\frac{1}{2} x) - mg (\frac{1}{2} = \frac{1}{2} m l + l + x)^2 - mgl mgx mgx^2 I \frac{1}{2} .2 2 21 2 21 2$ The loss of potential energy is therefore $--\frac{1}{2}mgx + --. 21$ This is equal to the gain in kinetic energy $--\frac{1}{2}my$, and therefore Another method: $v^2 : gx +--gx^2. 1 u -[-x l+x \frac{1}{2} oA$ Consider a point A. Anywhere will do, but I have chosen it to be a distance I below the level of the table and I to the left of the table edge. The moment of momentum (= angular momentum) of the chain about this point is $mlv = mlS$., and its rate of change is

therefore $m/5$. The torque about A is $(-)\frac{1}{2}l + X.)mgl = (Sl+x)mg$. These are equal, and $l dv$ so $l = g(-)\frac{1}{2}l + x)$. Write $5 = v$ in the usual way, and integrate (with $v = 0$ when $x dx = 0$) and the result $v^2 = gx + g--x^2$ follows. 1 To find the relation between x and t we can use the energy equation 9.2.9 for

conservative systems $\frac{1}{2}mv^2 + U(x) = E$ Here $x_0 = 0$ and we have already seen that $E - U(x) = \frac{1}{2}mv^2$ this expression, we obtain, after a little algebra and calculus, $\frac{1}{v} \frac{dv}{dx} = \frac{1}{2} \frac{d}{dx} \ln v^2$ The converse of this is the required expression $\ln(v^2) = \frac{1}{2} \int \frac{d}{dx} \ln v^2 dx = \frac{1}{2} \ln v^2 + C$ Upon integrating Differentiation of this with respect to time produces the third required expression: $\frac{d}{dt} \ln(v^2) = \frac{1}{v^2} \frac{dv^2}{dt} = \frac{1}{v^2} 2v \frac{dv}{dt} = 2 \frac{1}{v} \frac{dv}{dt}$ You may verify from these last two equations, if you wish, that $v^2 = g x + g x_0^2$.

35. (a) The maximum overhang of book 1 is $d_1 = w$. The centre mass of 1+2 is at $3w/2$ from the left hand side (LHS) of 2, so $d_2 = w/2$. The distance of the centre of mass of 1+2+3 is at $5w/2$ from the LHS of 3, so $d_3 = w/3$. Thus $D = d_1 + d_2 + d_3 = (1 + \frac{1}{2} + \frac{1}{3})w = 1.8w$. (b) In a similar manner we find that, given $n + 1$ books, the maximum overhang is $D = (1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n})w$. I don't know if there is a simple expression for the sum to n terms of this harmonic series. Please let me know if you know of one or can find one. Therefore I used a computer to solve $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} = 10$ by brute force. I got $n = 12367$, so you would need 12368 books. (c) The harmonic series is divergent and has no finite limit, so there is no finite limit to the possible overhang. You might wish to speculate on any practical limitations on constructing such a pile of books. For example, we have been assuming a uniform gravitational field - but this will no longer be valid once the overhang becomes comparable to the radius of Earth. This will, however, need quite a large number of books.